

GENERALIZED AND WEIGHTED STRICHARTZ ESTIMATES

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ABSTRACT. In this paper, we explore the relations between different kinds of Strichartz estimates and give new estimates in Euclidean space \mathbb{R}^n . In particular, we prove the generalized and weighted Strichartz estimates for a large class of dispersive operators including the Schrödinger and wave equation. As a sample application of these new estimates, we are able to prove the Strauss conjecture with low regularity for dimension 2 and 3.

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1. INTRODUCTION

In this paper, we explore the relations between different kinds of Strichartz estimates and give new estimates in Euclidean space \mathbb{R}^n . In particular, we prove the generalized and weighted Strichartz estimates for a large class of dispersive operators including the Schrödinger and wave equation. As a sample application of these new estimates, we are able to prove the Strauss conjecture with low regularity for dimension 2 and 3. In some sense, this paper can be viewed as a sequel to the work of Fang and the second author [5].

Let $D = \sqrt{-\Delta}$. Typically, Strichartz estimates for the dispersive operators e^{itD^a} , $a = 1, 2$, are a family of estimates which state

$$(1.1) \quad \|e^{itD^a} f\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^n)} \leq C \|f\|_{\dot{H}^s},$$

where \dot{H}^s ($s < n/2$) denotes the homogenous Sobolev space in \mathbb{R}^n . These estimates were first established by Strichartz [35] for $q = r$. They were generalized to non-endpoint admissible (q, r) by Ginibre and Velo [7] [8], Lindblad and Sogge [21]. The end point estimates were proven by Keel and Tao [18]. They are powerful tools in the study of the nonlinear Schrödinger and wave equations. See for example Cazenave [2], Sogge [29] and Tao [37] and references therein.

Let $\Delta_\omega = \sum_{1 \leq i < j \leq n} \Omega_{ij}^2$ be the Laplace-Beltrami operator on the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ with $\Omega_{ij} = x_i \partial_j - x_j \partial_i$, $\omega \in \mathbb{S}^{n-1}$, and define $\Lambda_\omega = \sqrt{1 - \Delta_\omega}$. Based on the usual Sobolev spaces H^s , we introduce the Sobolev spaces with angular regularity as follows ($b \geq 0$)

$$H_\omega^{s,b} = \Lambda_\omega^{-b} H^s = \{u \in H^s : \|\Lambda_\omega^b u\|_{H^s} < \infty\}.$$

For the homogeneous Sobolev space $\dot{H}^s = D^{-s} L^2$, we can similarly define the space $\dot{H}_\omega^{s,b} = \Lambda_\omega^{-b} D^{-s} L^2$. We will also use the homogeneous Besov space $\dot{B}_{p,q}^s$ (for $sp < n$ or $sp = n$ and $q = 1$), which is defined to be the completion of C_0^∞ in S' , with respect to the norm $\|f\|_{\dot{B}_{p,q}^s} = \|2^{sk} S_k f\|_{\ell_k^q L^p}$. Here S_k are the Fourier multiplier operators of the homogenous Littlewood-Paley decomposition.

Recently, there have been many works on various generalizations of the Strichartz estimates and their applications. Before stating our results and related works, we would like to list different types of Strichartz estimates by the following table. After tagging different Strichartz estimates, it will be easier to explain the history and

give an overview of our work by diagrams and lists afterward. Here, in general, we will be able to consider the operators e^{itD^a} with $a > 0$. Recall that the operators e^{itD^a} are related to the Schrödinger equation ($a = 2$) and the wave equation ($a = 1$). Also we denote $L_T^q L_x^r$ as $L_t^q L_x^r$ with domain $[0, T] \times \mathbb{R}^n$.

Table 1. Different types of Strichartz estimates.

Name	Left Norm	Right Norm	Range of q, r ($q, r \geq 2$)	No.
Strichartz	$L_t^q L_x^r$	$\dot{H}^s(H^s)$	$\frac{1}{q} \leq \frac{n-1}{2}(\frac{1}{2} - \frac{1}{r})$	(I)
G. Strichartz	$L_t^q L_x^r$	$\dot{H}_\omega^{s, 1/q}$	$\frac{1}{q} < (n-1)(\frac{1}{2} - \frac{1}{r})$	(II)
G. Strichartz	$L_t^q L_{ x }^r L_\omega^2$	\dot{H}^s	$\frac{1}{q} < (n-1)(\frac{1}{2} - \frac{1}{r})$	(III)
Loc. G. Strichartz	$L_T^q L_x^r$	$\dot{H}_\omega^{s, b}$	$(n-1)(\frac{1}{2} - \frac{1}{r}) \leq \frac{1}{q}$	(IV)
Loc. G. Strichartz	$L_T^q L_{ x }^r L_\omega^2$	\dot{H}^s	$(n-1)(\frac{1}{2} - \frac{1}{r}) \leq \frac{1}{q}$	(V)
Morawetz-KSS	$\langle x \rangle^\alpha L_{T,x}^2$	L_x^2	$\alpha \geq 0$	(VI)
W. Strichartz	$ x ^\alpha L_t^q L_{ x }^r L_\omega^2$	$\dot{H}_\omega^{s, b}$	$\frac{1}{q} - \frac{n-1}{2} + \frac{n-1}{r} < \alpha < \frac{n}{r}$	(VII)
W. Strichartz	$ x ^\alpha L_t^q L_x^r$	$\dot{H}_\omega^{s, b}$	$\frac{1}{q} - \frac{n-1}{2} + \frac{n-1}{r} < \alpha < \frac{n}{r}$	(VIII)

Now let us give a brief history of the generalized Strichartz estimates (G. Strichartz) and weighted Strichartz estimates (W. Strichartz) within our best knowledge. The generalized Strichartz estimates were first studied in the endpoint case of the classical Strichartz estimates for the wave and Schrödinger equations. For the wave equation ($a = 1$), it is known that the 3-dimensional endpoint $L_t^2 L_x^\infty$ Strichartz estimate fails ([19]), however, the corresponding generalized estimates (II) and (III) were proven in [23]. For the Schrödinger equation ($a = 2$), Montgomery-Smith [26] proved the failure of the $L_t^2 L_x^\infty$ Strichartz estimate and Tao [36] proved the 2-dimensional endpoint $L_t^2 L_{|x|}^\infty L_\omega^2$ estimate.

Then, for the wave equation, generalized Strichartz estimates of type (II) were proven in Sterbenz [33] ($n \geq 3$) and Fang and Wang [5] ($n \geq 2$). When the initial data is radial, the localized estimates (IV) have also been obtained in Hidano-Kurokawa [12], by proving certain weighted radial Hardy-Littlewood-Sobolev estimates. For the 2-dimensional wave equation, the generalized Strichartz estimates of type (III) and (V) were proven recently by Smith, Sogge and Wang [27] and Fang and Wang [6] respectively.

Around the same time, Keel, Smith and Sogge [17] proved the estimates (VI) for the wave equation when $\alpha = 1/2$, $n = 3$, which were named KSS estimates or Morawetz-KSS estimates. The estimates of this type were developed drastically afterwards (see e.g. [24], [15], [25], [30] and [14]).

In some sense, the Morawetz-KSS estimates can be viewed as a special case of the weighted Strichartz estimates (VII) and (VIII). The weighted Strichartz estimates (VII) with $q = r$ were proven by Fang and Wang [5] ($a > 0$) and Hidano, Metcalfe, Smith, Sogge and Zhou [13] ($a = 1$), with the previous work for $a = 1$ and radial data in [10].

As was clear from [5] and [13], these estimates are intimately related with each other. A starting point can be the homogenous trace lemma (H.T.L., see (1.3) of

[5]), i.e.,

$$(1.2) \quad r^{(n-b)/2} \|f(r\omega)\|_{L_\omega^2} \leq C \|D^{b/2} \Lambda_\omega^{(1-b)/2} f\|_{L_x^2}, \quad b \in (1, n).$$

Using this and interpolation, we can conclude the case $q = r$ in (VII) as follows,

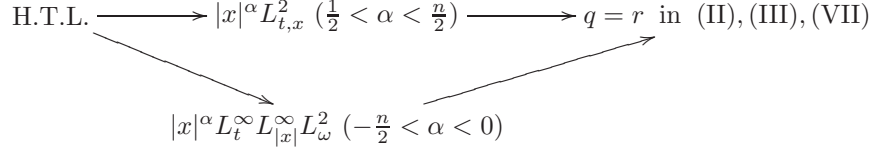


Figure 1. Weighted Strichartz estimates from H.T.L..

Similarly, for the wave equation $a = 1$, by using the inhomogenous trace lemma (I.T.L., see (1.7) in [5]), i.e.,

$$(1.3) \quad r^{(n-1)/2} \|f(r\omega)\|_{L_\omega^2} \leq C \|f\|_{\dot{B}_{2,1}^{1/2}},$$

we can conclude a couple of estimates in (VI), (II) and (VIII) (see [33] for the Rodnianski's argument to deduce (II) from (VI.a)).

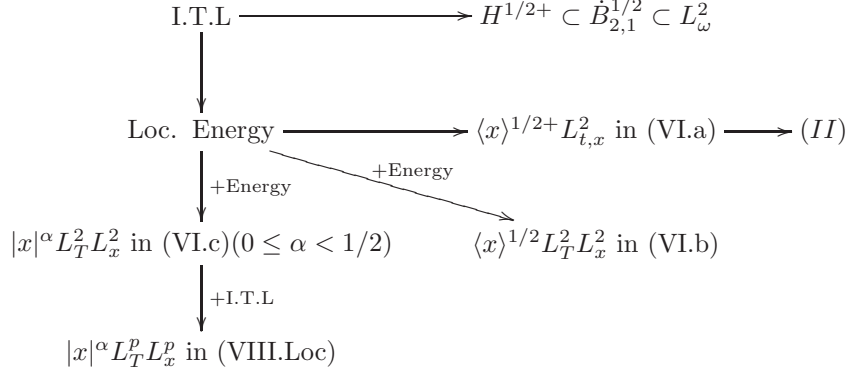


Figure 2. Some consequences of I.T.L..

As long as the estimates in the above diagrams were built, we can get our results as follows.

- Use (VII) with $q = r$ ([5]) and Rodnianski's argument, we can get estimates in (II) for $n \geq 2$ and $a > 0$, i.e. Theorem 1.1.
- Use (VI.b), (VI.c) and Rodnianski's argument, to get estimates in (IV) with $a = 1$, i.e. Theorem 1.3.
- Use the arguments of [27] and [6], to get estimates in (III) and (V) with $a = 1$, i.e. Theorem 1.4.
- Use the argument of Sterbenz [33], to get estimates in (VIII) for $a = 1$ and radial functions, i.e. Theorem 1.5.
- Use $q = r$ in (VII), Rodnianski's argument, together with a localized version of the weighted Hardy-Littlewood-Sobolev inequality, to get estimates in the range of $q \leq r$ in (VIII), i.e. Theorem 1.6.

- Use $q = r$ in (VI) and Hardy's inequality, to get estimates in the range of $q \geq r$ in (VII), i.e. Theorem 1.7.

We now state our results precisely. First we would like to give an angular generalization of the classical Strichartz estimates.

Theorem 1.1 (Generalized Strichartz Estimates). *Let $n \geq 2$, $a > 0$, $q, r \geq 2$ and $r < \infty$. If*

$$\frac{1}{q} < (n-1) \left(\frac{1}{2} - \frac{1}{r} \right) \text{ or } (q, r) = (\infty, 2),$$

then we have

$$(1.4) \quad \|e^{itD^a} f\|_{L_t^q L_x^r} \lesssim \|f\|_{\dot{H}_\omega^{s, \frac{1}{q}}}$$

with $s = n(\frac{1}{2} - \frac{1}{r}) - \frac{a}{q}$.

Remark 1.1. The technical restriction $r < \infty$ can essentially be removed (except the endpoint $(q, r) = (\infty, \infty), (2, \infty)$), if we use the real interpolation argument as in Section 5.2.

Remark 1.2. In the case of the wave equation ($a = 1$), recall that we have the classical Strichartz estimates (see e.g. [4], [18])

$$\|e^{itD} f\|_{L_t^q L_x^r} \lesssim \|f\|_{\dot{H}^s},$$

under the admissible condition

$$(1.5) \quad \frac{1}{q} \leq \min \left(\frac{1}{2}, \frac{n-1}{2} \left(\frac{1}{2} - \frac{1}{r} \right) \right), (q, r) \neq (\max(2, \frac{4}{n-1}), \infty), (q, r) \neq (\infty, \infty).$$

The result in Theorem 1.1 extends the Strichartz estimates to the case of

$$\frac{1}{q} < (n-1) \left(\frac{1}{2} - \frac{1}{r} \right)$$

by requiring some additional angular regularity on the data.

Remark 1.3. The requirement for (q, r) are sharp for $a = 1$, see Remark 1.7. When $a \neq 1$, the sharpness may be different. It will be interesting to determine the sharp range for (q, r) , at least for the Schrödinger equation ($a = 2$). Recall that for the Schrödinger equation ($a = 2$), the Strichartz estimates can be stated as follows (see e.g. [18], (1.26) of [5])

$$(1.6) \quad \|e^{it\Delta} f\|_{L_t^q L_x^r} \lesssim \|f\|_{\dot{H}^s},$$

for

$$(1.7) \quad \frac{1}{q} \leq \min \left(\frac{1}{2}, \frac{n}{2} \left(\frac{1}{2} - \frac{1}{r} \right) \right), (q, r) \neq (\infty, \infty), (2, \infty).$$

We note here that for $n = a = 2$, our estimate is worse than the standard one.

In fact, for the wave equation ($a = 1$), we can improve the required angular regularity to be almost optimal for the non-admissible (q, r) , by interpolating with the classical Strichartz estimates, which recover the results in [33] for $n \geq 3$ and [5] for the full range $n \geq 2$.

Corollary 1.2. *Let $n \geq 2$,*

$$s = n(\frac{1}{2} - \frac{1}{r}) - \frac{1}{q}, \quad s_{kn} = \frac{2}{q} - (n-1)(\frac{1}{2} - \frac{1}{r}),$$

and

$$(1.8) \quad \frac{n-1}{2}(\frac{1}{2} - \frac{1}{r}) < \frac{1}{q} < (n-1)(\frac{1}{2} - \frac{1}{r}), \quad q \geq 2.$$

Then we have the estimates

$$(1.9) \quad \|e^{itD} f\|_{L_t^q L_x^r} \lesssim \|f\|_{\dot{H}_\omega^{s,b}}$$

for any $b > s_{kn}$.

On the other hand, if we localize the domain in finite time interval $[0, T]$, by making use of KSS estimates, we will get the following localized Strichartz estimates for the wave equation.

Theorem 1.3. *Let $n \geq 2$, $2 \leq r < \infty$ and*

$$\frac{1}{q} = (n-1)(\frac{1}{2} - \frac{1}{r}) \leq \frac{1}{2}.$$

Then we have

$$(1.10) \quad \|e^{itD} f\|_{L_T^q L_x^r} \lesssim (\ln(2+T))^{1/q} \|(\ln(2+D))^{1/q} D^{\frac{1}{2}-\frac{1}{r}} \Lambda_\omega^{1/q} f\|_{L_x^2}.$$

For the endpoint case $(q, r, n) = (2, \infty, 2)$ and any $\epsilon > 0$, we have

$$(1.11) \quad \|e^{itD} f\|_{L_T^2 L_x^\infty} \lesssim (\ln(2+T))^{\frac{1}{2}} \|f\|_{H_\omega^{\frac{1}{2}+\epsilon, \frac{1}{2}+\epsilon}}.$$

Moreover, if $(n-1)(\frac{1}{2} - \frac{1}{r}) < \frac{1}{q} \leq \frac{1}{2}$, we have

$$(1.12) \quad \|e^{itD} f\|_{L_T^q L_x^r} \lesssim T^{\frac{1}{q} - (n-1)(\frac{1}{2} - \frac{1}{r})} \|f\|_{\dot{H}_\omega^{s,b}},$$

with $s = \frac{1}{2} - \frac{1}{r}$ and $b = (n-1)(\frac{1}{2} - \frac{1}{r})$.

In the recent work of Smith, Sogge and Wang [27] (see also Fang and Wang [6]), we see that when $n = 2$, we can in fact improve further the generalized Strichartz estimates for the wave equation to the $L_t^q L_{|x|}^r L_\omega^2$ estimates, in which case, the angular regularity is not required. Here $L_{|x|}^r$ denotes the Lebesgue space for the variable $|x|$ with respect to the measure $|x|^{n-1} d|x|$. Inspired by their work, we generalize the results to the general spatial dimensions.

Theorem 1.4. *Let $n \geq 2$ and $q, r \geq 2$. If*

$$\frac{1}{q} < (n-1)(\frac{1}{2} - \frac{1}{r}) \text{ or } (q, r) = (\infty, 2), (q, r) \neq (2, \infty), (q, r) \neq (\infty, \infty),$$

then we have

$$(1.13) \quad \|e^{itD} f\|_{L_t^q L_{|x|}^r L_\omega^2} \lesssim \|f\|_{\dot{H}^s}$$

with $s = \frac{n}{2} - \frac{n}{r} - \frac{1}{q}$. On the other hand, if $s = \frac{1}{2} - \frac{1}{r}$ and $\frac{1}{q} > (n-1)(\frac{1}{2} - \frac{1}{r})$, then

$$(1.14) \quad \|e^{itD} f\|_{L_T^q L_{|x|}^r L_\omega^2} \lesssim T^{\frac{1}{q} - (n-1)(\frac{1}{2} - \frac{1}{r})} \|f\|_{\dot{H}^s}.$$

Moreover, if $\frac{1}{q} = (n-1)(\frac{1}{2} - \frac{1}{r})$, then for any $\delta > 0$, we have

$$(1.15) \quad \|e^{itD} f\|_{L_T^q L_{|x|}^r L_\omega^2} \lesssim (\ln(2+T))^{1/q} \|f\|_{H_x^{\frac{1}{2} - \frac{1}{r} + \delta}}.$$

Remark 1.4. As a complement, we cite the endpoint Strichartz estimates when $n = 2$ here. For $2 < q < \infty$, Smith, Sogge and Wang [27] prove that

$$(1.16) \quad \|e^{itD} f\|_{L_t^q L_{|x|}^\infty L_\omega^2(\mathbb{R} \times \mathbb{R}^2)} \leq C_q \|f\|_{\dot{H}^\gamma(\mathbb{R}^2)}, \quad \gamma = 1 - 1/q.$$

Moreover, Fang and Wang [6] prove the endpoint estimates for $q = 2$,

$$(1.17) \quad \|e^{itD} f\|_{L_t^2 L_{|x|}^\infty L_\omega^2([0, T] \times \mathbb{R}^2)} \leq C_\gamma (\ln(2 + T))^{\frac{1}{2}} \|f\|_{H^\gamma(\mathbb{R}^2)}$$

for any $\gamma > 1/2$.

Remark 1.5. For the frequency localized functions, the estimate (1.13) holds for any $q, r \geq 2$ such that $\frac{1}{q} < (n-1)(\frac{1}{2} - \frac{1}{r})$ (see (3.1)).

Remark 1.6. The same type of estimate to (1.13) was proved in [23] for $(q, r, n) = (2, \infty, 3)$. We also remark that a similar estimate was proved for the Klein-Gordon equation and Schrödinger equation with $(q, r, n) = (2, \infty, 2)$ in [16].

Next, we are interested in exploiting the weighted Strichartz estimates for any $q, r \geq 2$. The first result is for the wave equation and radial initial data, which will serve as a guideline for the general estimates.

Theorem 1.5 (Weighted Strichartz estimates for radial initial data). *Let $q, r \geq 2$, and u be a radial function on \mathbb{R}^{n+1} such that $\square u = (\partial_t^2 - \Delta)u = 0$. Then the following estimates hold with $s = \alpha + n(\frac{1}{2} - \frac{1}{r}) - \frac{1}{q}$:*

$$(1.18) \quad \| |x|^{-\alpha} u \|_{L_t^q L_x^r} \lesssim \|u(0)\|_{\dot{H}^s} + \|\partial_t u(0)\|_{\dot{H}^{s-1}},$$

when

$$\begin{cases} \frac{1}{q} - (n-1)(\frac{1}{2} - \frac{1}{r}) < \alpha < \frac{n}{r}, & 2 \leq q, r < \infty, \\ -(n-1)(\frac{1}{2} - \frac{1}{r}) \leq \alpha < \frac{n}{r} & q = \infty, 2 \leq r < \infty, \\ \frac{1}{q} - \frac{n-1}{2} < \alpha \leq 0 & r = \infty, 2 < q < \infty. \end{cases}$$

Remark 1.7. The requirement on α is essentially optimal. In fact, since the decay estimates for the wave equation are sharp in general even for radial functions (see e.g. Lemma 4.1 of [12]). By those estimates, it is easy to see that to bound the left hand side of (1.18), we must have

$$\begin{aligned} \frac{1}{q} - \frac{n-1}{2} + \frac{n-1}{r} &< \alpha < \frac{n}{r}, \quad q, r < \infty, \\ \frac{1}{q} - \frac{n-1}{2} &< \alpha \leq 0, \quad q < r = \infty, \\ -\frac{n-1}{2} + \frac{n-1}{r} &\leq \alpha < \frac{n}{r}, \quad r < q = \infty. \end{aligned}$$

The second result is the weighted Strichartz estimates for general $a > 0$ and general data.

Theorem 1.6. *Let*

$$(1.19) \quad 2 \leq q \leq r < \infty, \text{ and } \frac{1}{q} - \frac{n-1}{2} + \frac{n-1}{r} < \alpha < \frac{n}{r}.$$

Then we have the following weighted Strichartz estimates,

$$(1.20) \quad \| |x|^{-\alpha} e^{itD^a} f \|_{L_t^q L_x^r} \lesssim \|f\|_{\dot{H}_\omega^{s,b}},$$

where

$$-\alpha + \frac{a}{q} + \frac{n}{r} = -s + \frac{n}{2}, \quad b = \frac{1}{q} - \alpha.$$

In Theorem 1.6, we have an additional restriction $q \leq r$ for q, r , compared with (1.18). In general, we can relax this restriction.

Theorem 1.7. *Let $2 \leq r \leq q \leq \infty$, and $\frac{1}{q} - \frac{n-1}{2} + \frac{n-1}{r} < \alpha < \frac{n}{r}$. Then we have the following weighted Strichartz estimates,*

$$(1.21) \quad \| |x|^{-\alpha} e^{itD^a} f \|_{L_t^q L_{|x|}^r L_\omega^2} \lesssim \| f \|_{\dot{H}_\omega^{s,b}},$$

where

$$-\alpha + \frac{a}{q} + \frac{n}{r} = -s + \frac{n}{2}$$

and

$$\begin{cases} b \geq -\alpha + \frac{1}{q} - (n-1)\left(\frac{1}{2} - \frac{1}{r}\right), & \text{if } \frac{1}{q} - \frac{n-1}{2} + \frac{n-1}{r} < \alpha < \frac{n}{r}, \\ b > -\frac{n-1}{q} - (n-1)\left(\frac{1}{2} - \frac{1}{r}\right), & \text{if } \frac{n}{q} \leq \alpha < \frac{n}{r}. \end{cases}$$

Remark 1.8. The results in Theorem 1.6 and Theorem 1.7 are generalizations of the weighted Strichartz estimates in Fang and Wang [5] (see also [13] for $a = 1$), i.e.

$$(1.22) \quad \| |x|^{\frac{a}{2} - \frac{n}{r} - \frac{b}{2}} e^{itD^a} f(x) \|_{L_{t,|x|}^r L_\omega^2} \lesssim \| D^{\frac{b}{2} - \frac{a}{r}} \Lambda_\omega^{\frac{1-b}{2}} f \|_{L_x^2},$$

if $b \in (1, n)$ and $r \in [2, \infty]$. In particular, when $r = 2$, we have the generalized Morawetz estimates

$$(1.23) \quad \| |x|^{-\frac{b}{2}} e^{itD^a} f \|_{L_{t,x}^2} \lesssim \| D^{\frac{b-a}{2}} \Lambda_\omega^{\frac{1-b}{2}} f \|_{L_x^2},$$

for any $b \in (1, n)$ and $a > 0$.

Lastly, we present local in time weighted Strichartz estimates for the wave equation.

Theorem 1.8. *Let $n \geq 2$, $2 \leq p < \infty$, and $(n-1)(\frac{1}{p} - \frac{1}{2}) < \alpha < \frac{n}{p} - \frac{n-1}{2}$, then we have the following weighted Strichartz estimates,*

$$(1.24) \quad \| |x|^{-\alpha} e^{itD} f \|_{L_t^p L_{|x|}^p L_\omega^2} \lesssim T^{-\alpha - \frac{n-1}{2} + \frac{n}{p}} \| f \|_{\dot{H}^s},$$

where $s = \frac{1}{2} - \frac{1}{p}$.

This theorem comes from an interpolation between KSS estimates and the inhomogenous trace lemma (1.3), as in [41].

Remark 1.9. We can also get more general estimates, if we interpolate the KSS estimates with the weighted Strichartz estimates (1.22) with $r = \infty$.

This paper is arranged as follows. In section 2 we prove the estimates stated in Theorem 1.1 and Theorem 1.3; In section 3 we prove Theorem 1.4; In section 4 we prove Theorem 1.5; In section 5 we prove Theorem 1.6 and 1.7; Lastly we provide an application of the Strichartz estimates in Section 6.

2. GENERALIZED STRICHARTZ ESTIMATES

In this section we prove Theorem 1.1, from which we see how the generalized Strichartz estimates can be obtained from the weighted Strichartz estimates. We also prove Theorem 1.3 which illustrates that local in time generalized Strichartz estimates can be obtained from the Morawetz-KSS estimates.

2.1. Generalized Strichartz Estimates. Now we prove Theorem 1.1 by using weighted Strichartz estimate (1.22) and Rodnianski's argument (see [33]).

Let $f_{1,N}$ be a unit frequency function of angular frequency N and $u_{1,N} = e^{itD^a} f_{1,N}$ with $a > 0$. Denote the norm

$$\|f\|_{\ell^p L^q_Q} = \left(\sum_{\alpha} \|f\|_{L^q(Q_{\alpha})}^p \right)^{\frac{1}{p}},$$

where $\{Q_{\alpha}\}$ is a partition of \mathbb{R}^n into cubes Q_{α} of side length 1.

First, since $q \geq 2$, by using the Sobolev embedding $\langle N \rangle^{-\frac{n-1}{q}} L^q_{\omega} \subset L^{\infty}_{\omega}$ on the unit sphere \mathbb{S}^{n-1} for angular frequency localized functions (see Lemma 7.2 in Appendix 7), we have the following estimate for any tiling of \mathbb{R}^n by cubes $\{Q_{\alpha}\}$ of side length 1:

$$\begin{aligned} \|u_{1,N}(t)\|_{L^q(Q_{\alpha})} &\lesssim (\|1\|_{L^q_{\omega}(|x|\omega \in Q_{\alpha})} \|u_{1,N}(t)\|_{L^{\infty}_{\omega}})_{L^q_{|x|}} \\ &\lesssim \|\langle x \rangle^{-\frac{n-1}{q}} \langle N \rangle^{\frac{n-1}{q}} u_{1,N}(t)\|_{L^q_x(Q_{\alpha})}, \end{aligned}$$

where we have used the fact that

$$|\{\omega : |x|\omega \in Q_{\alpha}\}| \lesssim (1 + |x|)^{-(n-1)}.$$

This means that we have

$$(2.1) \quad \|u_{1,N}(t)\|_{\ell^{\infty} L^q_Q} \lesssim \|\langle x \rangle^{-\frac{n-1}{q}} \langle N \rangle^{\frac{n-1}{q}} u_{1,N}(t)\|_{L^q_x}.$$

Interpolating this with the trivial estimate

$$\|u_{1,N}(t)\|_{\ell^q L^q_Q} \lesssim \|u_{1,N}(t)\|_{L^q_x},$$

we arrive at the following estimate for $r \geq q$:

$$(2.2) \quad \|u_{1,N}\|_{L^q_t \ell^r L^q_Q} \lesssim \|\langle x \rangle^{-d} \langle N \rangle^d u_{1,N}\|_{L^q_t L^q_x},$$

where $d = (n-1)(\frac{1}{q} - \frac{1}{r})$.

Recall the weighted Strichartz estimates (1.22), i.e.,

$$(2.3) \quad \||x|^{\frac{n}{2} - \frac{n}{q} - \frac{b}{2}} e^{itD^a} f(x)\|_{L^q_{t,|x|} L^2_{\omega}} \lesssim \|D^{\frac{b}{2} - \frac{a}{q}} \Lambda_{\omega}^{\frac{1-b}{2}} f\|_{L^2_x},$$

if $b \in (1, n)$, $a > 0$ and $q \in [2, \infty]$. By Sobolev embedding on the sphere \mathbb{S}^{n-1} (see (7.4) in Appendix 7), we have

$$(2.4) \quad \||x|^{\frac{n}{2} - \frac{n}{q} - \frac{b}{2}} e^{itD^a} f_{1,N}(x)\|_{L^q_{t,x}} \lesssim \|\langle N \rangle^{\frac{n-b}{2} - \frac{n-1}{q}} f_{1,N}\|_{L^2_x}.$$

Combining (2.2) and (2.4), we have

$$(2.5) \quad \|e^{itD^a} f_{1,N}\|_{L^q_t \ell^r L^q_Q} \lesssim \|\langle N \rangle^{1/q} f_{1,N}\|_{L^2_x} \sim \|\Lambda_{\omega}^{1/q} f_{1,N}\|_{L^2_x},$$

if $1/q < (n-1)(1/2 - 1/r)$, $2 \leq q \leq r$.

We can see that (2.5) allows a wider range of (q, r) than that in the usual Strichartz estimates, i.e., we have

$$(2.6) \quad \|e^{itD^a} f_{1,N}\|_{L_t^q L_x^r} \lesssim \|\Lambda_\omega^{1/q} f_{1,N}\|_{L_x^2}.$$

To see this, we compute, using the Sobolev embedding in Q_α , that for any $q \leq r$:

$$\begin{aligned} \|e^{itD^a} f_{1,N}\|_{L_t^q L_x^r} &= \|e^{itD^a} f_{1,N}\|_{L_t^q \ell_\alpha^r L^r(Q_\alpha)} \\ &\lesssim \|e^{itD^a} (1-\Delta)^n f_{1,N}\|_{L_t^q \ell_\alpha^r L^r(Q_\alpha)} \\ &\lesssim \|\Lambda_\omega^{1/q} (1-\Delta)^n f_{1,N}\|_{L_x^2} \\ &\lesssim \|\Lambda_\omega^{1/q} f_{1,N}\|_{L_x^2}. \end{aligned}$$

Now we do the Littlewood-Paley decomposition $f = \sum_{j \in \mathbb{Z}} f_j$, where $\hat{f}_j(\xi) = (\varphi(\xi/2^j) - \varphi(\xi/2^{j+1}))\hat{f}(\xi)$ for some real-valued radially symmetric bump function $\varphi(\xi)$ adapted to $\{\xi \in \mathbb{R}^n : |\xi| \leq 2\}$ which equals 1 on the unit ball. Furtherly we make a spherical decomposition of each f_j and let $f_j = \sum_{N \in 2^{\mathbb{N}} \cup 0} f_{jN}$, where f_{jN} has angular frequency N . By using Littlewood-Paley-Stein theorem (see Theorem 2 [34]) and applying (2.6), we get for $r < \infty$

$$\begin{aligned} \|e^{itD^a} f\|_{L_t^q L_x^r} &\simeq \|e^{itD^a} f_{jN}\|_{L_t^q L_x^r l_{j,N}^2} \\ &\lesssim \|e^{itD^a} f_{jN}\|_{l_{j,N}^2 L_t^q L_x^r} \\ &\lesssim \|2^{js} \Lambda_\omega^{1/q} f_{jN}\|_{l_{j,N}^2 L_x^2} \quad \text{where } s = \frac{n}{2} - \frac{a}{q} - \frac{n}{r} \\ &\lesssim \|2^{js} \Lambda_\omega^{1/q} f_j\|_{l_j^2 L_x^2} \\ &\simeq \|\Lambda_\omega^{1/q} f\|_{\dot{H}^s}. \end{aligned}$$

This completes the proof of Theorem 1.1 for $2 \leq q \leq r$ and $r < \infty$. The case when $q \geq r$ comes from interpolation with the energy estimate with $(q, r) = (\infty, 2)$.

2.2. Local in Time Strichartz Estimates for the Wave Equation. In this subsection, we prove Theorem 1.3.

When considering the wave equation, if we denote

$$A_\mu(T) = \begin{cases} 1, & \mu > 1/2, \\ \log(2+T)^{\frac{1}{2}}, & \mu = 1/2, \\ T^{\frac{1}{2}-\mu}, & 0 \leq \mu < 1/2, \end{cases}$$

then the Morawetz-KSS estimates can be stated as

$$(2.7) \quad \|\langle x \rangle^{-\mu} e^{itD} f\|_{L_{[0,T]}^2 L_x^2} \lesssim A_\mu(T) \|f\|_{L_x^2}.$$

For the sake of completeness, we present the proof of the Morawetz-KSS estimates in Appendix 7.2.

We consider now the remaining case $1/q \geq (n-1)(1/2 - 1/r)$ and $q, r \geq 2$. We will apply the Morawetz-KSS estimates for the wave equation to conclude some local in time generalized Strichartz estimates for $a = 1$.

Proof of Theorem 1.3. Set $q = 2$ in (2.2), if $d = 1/2$ (and hence $r = 2\frac{n-1}{n-2}$), we have by (2.7)

$$\begin{aligned} \|e^{itD} f_{1,N}\|_{L_T^2 L_x^r} &= \|e^{itD} f_{1,N}\|_{L_T^2 \ell^r L_Q^r} \\ &\lesssim \|e^{itD} (1 - \Delta)^n f_{1,N}\|_{L_T^2 \ell^r L_Q^2} \\ &\lesssim \|\langle x \rangle^{-1/2} N^{1/2} e^{itD} (1 - \Delta)^n f_{1,N}\|_{L_T^2 L_x^2} \\ &\lesssim (\ln(2 + T))^{1/2} \|N^{1/2} (1 - \Delta)^n f_{1,N}\|_{L_x^2} \\ &\lesssim (\ln(2 + T))^{1/2} \|\Lambda_\omega^{1/2} f_{1,N}\|_{L_x^2}. \end{aligned}$$

Interpolating with the energy estimates, we have

$$(2.8) \quad \|e^{itD} f_{1,N}\|_{L_T^q L_x^r} \lesssim (\ln(2 + T))^{1/q} \|\Lambda_\omega^{1/q} f_{1,N}\|_{L_x^2},$$

for $1/q = (n - 1)(1/2 - 1/r) \leq 1/2$.

By rescaling, we have that for any $\lambda > 0$,

$$\begin{aligned} \|e^{itD} f_{\lambda,N}\|_{L_T^q L_x^r} &\lesssim (\ln(2 + \lambda T))^{1/q} \lambda^{1/2-1/r} \|\Lambda_\omega^{1/q} f_{\lambda,N}\|_{L_x^2} \\ &\lesssim (\ln(2 + T))^{1/q} (\ln(2 + \lambda))^{1/q} \lambda^{1/2-1/r} \|\Lambda_\omega^{1/q} f_{\lambda,N}\|_{L_x^2}. \end{aligned}$$

Then (1.10) and (1.11) come from the Littlewood-Paley decomposition.

For the case $(n - 1)(1/2 - 1/r) < 1/q \leq 1/2$, we first set $q = 2$ in (2.2), then for $d < 1/2$ (and hence $r < 2\frac{n-1}{n-2}$) we have

$$\|e^{itD} f_{1,N}\|_{L_T^2 \ell^r L_Q^2} \lesssim \|\langle x \rangle^{-d} N^d e^{itD} f_{1,N}\|_{L_T^2 L_x^2} \lesssim T^{1/2-d} \|\Lambda_\omega^d f_{1,N}\|_{L_x^2}.$$

Interpolating with the energy estimates, we get that

$$(2.9) \quad \|e^{itD} f_{1,N}\|_{L_T^q L_x^r} \lesssim T^{1/q-(n-1)(1/2-1/r)} \|\Lambda_\omega^{(n-1)(1/2-1/r)} f_{1,N}\|_{L_x^2},$$

for $(n - 1)(1/2 - 1/r) < 1/q \leq 1/2$. Again by rescaling and the Littlewood-Paley inequality we get (1.12).

3. $L^q L^r L_\omega^2$ GENERALIZED STRICHARTZ ESTIMATES

In this section, we give the proof of Theorem 1.4, inspired by the recent work of Smith, Sogge and Wang [27] and Fang and Wang [6].

We shall show that

$$(3.1) \quad \|e^{itD} f\|_{L_t^q L_{|x|}^r L_\omega^2(\mathbb{R} \times \mathbb{R}^n)} \leq C_{q,r,n} \|f\|_{L^2(\mathbb{R}^n)}$$

if $q \geq 2$, $1/q < (n - 1)(1/2 - 1/r)$ or $(q, r) = (\infty, 2)$, and \hat{f} is supported in $\{\xi : |\xi| \in [1/2, 1]\}$. We shall also prove that

$$(3.2) \quad \|e^{itD} f\|_{L_T^q L_{|x|}^r L_\omega^2(\mathbb{R} \times \mathbb{R}^n)} \leq C_{q,r}(T) \|f\|_{L^2(\mathbb{R}^n)}$$

for any $q, r \geq 2$, $1/q \geq (n - 1)(1/2 - 1/r)$, f such that $\text{supp } \hat{f} \subset \{\xi : |\xi| \in [1/2, 1]\}$ and

$$C_{q,r}(T) = \begin{cases} C_{q,r,n} T^{\frac{1}{q} - (n-1)(\frac{1}{2} - \frac{1}{r})} & \frac{1}{q} > (n-1)(\frac{1}{2} - \frac{1}{r}) \\ C_{q,r,n} (\ln(2 + T))^{\frac{1}{q}} & \frac{1}{q} = (n-1)(\frac{1}{2} - \frac{1}{r}) \end{cases}$$

By scaling, Littlewood-Paley theory and interpolation, we get from (3.1) that if we remove the support assumptions on the Fourier transform, then

$$(3.3) \quad \|e^{itD}g\|_{L_t^q L_{|x|}^r L_\omega^2(\mathbb{R} \times \mathbb{R}^n)} \leq C_{q,r,n} \|g\|_{\dot{H}^{n(1/2-1/r)-1/q}(\mathbb{R}^n)}.$$

for (q, r, n) satisfying

$$\frac{1}{q} < (n-1)(\frac{1}{2} - \frac{1}{r}) \text{ or } (q, r) = (\infty, 2), q \geq 2, (q, r) \neq (2, \infty), (q, r) \neq (\infty, \infty).$$

At first, we use scaling and Littlewood-Paley decomposition to conclude from (3.1) that we have

$$(3.4) \quad \|e^{itD}g\|_{L_t^q L_{|x|}^r L_\omega^2(\mathbb{R} \times \mathbb{R}^n)} \leq C_{q,r} \|g\|_{\dot{B}_{2,1}^{n(1/2-1/r)-1/q}(\mathbb{R}^n)}.$$

Recall that we have the interpolation between spaces of vector-valued functions (see 5.8.6 of [1] page 130 or Theorem 1.18.4 of [39] page 128)

$$(3.5) \quad (L^{q_0}(A_0), L^{q_1}(A_1))_{\theta,q} = L^q((A_0, A_1)_{\theta,q})$$

if $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, $1 \leq q_0, q_1 < \infty$ and $\theta \in (0, 1)$, and the interpolation of the homogeneous Besov spaces (see Theorem 6.4.5 of [1] page 152)

$$(3.6) \quad (\dot{B}_{2,1}^{s_0}, \dot{B}_{2,1}^{s_1})_{\theta,q} = \dot{B}_{2,q}^s \supset \dot{B}_{2,2}^s = \dot{H}^s$$

if $q \geq 2$.

Based on (3.4), (3.5) and (3.6) for fixed $r \in (2, \infty]$, we get for q with $0 < 1/q < (n-1)(1/2 - 1/r)$ and $q > 2$,

$$(3.7) \quad \|e^{itD}g\|_{L_t^q L_{|x|}^r L_\omega^2(\mathbb{R} \times \mathbb{R}^n)} \leq C_{q,r} \|g\|_{\dot{B}_{2,q}^{n(1/2-1/r)-1/q}(\mathbb{R}^n)} \leq C_{q,r} \|g\|_{\dot{H}^{n(1/2-1/r)-1/q}(\mathbb{R}^n)}.$$

This gives us the result (3.3) for $2 < q < \infty$. For the case $q = \infty$ and $r < \infty$, the result is just the consequence of energy estimates and Sobolev embedding. To prove the remaining case with $q = 2$ and $r < \infty$, we need only to use the fact that

$$(L_{|x|}^{r_1} L_\omega^2, L_{|x|}^{r_2} L_\omega^2)_{\theta,r} = L_{|x|}^r L_\omega^2$$

for $1 \leq r_1, r_2 < \infty$, $\theta/r_1 + (1-\theta)/r_2 = 1/r$ and $\theta \in (0, 1)$ (see Theorem 1.18.5 of [39] page 130).

This concludes the proof of (3.3) for $(q, r) \neq (2, \infty), (\infty, \infty)$ with $1/q < (n-1)(1/2 - 1/r)$ and $q \geq 2$. Also we can get (1.14) and (1.15) from (3.2) by the same argument.

Remark 3.1. All of the requirements for (q, r) are necessary for the estimates except the requirement $(q, r) \neq (2, \infty)$. In general, we expect that this restriction can be relaxed. In particular, when $n = 3$, the estimate with $(q, r) = (2, \infty)$ are proven to be true in Machihara, Nakamura, Nakanishi, and Ozawa [23]. However, we will not exploit this issue.

3.1. Proof of (3.1). To begin, let us recall some basic knowledge about the spherical harmonics (for detailed discussion, see e.g. Stein and Weiss [32]). Let $n \geq 2$. For any $k \geq 0$, we denote by \mathcal{H}_k the space of spherical harmonics of degree k on \mathbb{S}^{n-1} , by $d(0) = 1$ and $d(k) = \frac{2k+n-2}{k} C_{k-1}^{n+k-3} \simeq \langle k \rangle^{n-2}$ (for $k \geq 1$) its dimension, and by $\{Y_{k,1}, \dots, Y_{k,d(k)}\}$ the orthonormal basis of \mathcal{H}_k . It is well known that $L^2(\mathbb{S}^{n-1}) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k$ and that $F(t, x) = F(t, r\omega)$ has the expansion

$$(3.8) \quad F(t, r\omega) = \sum_{k=0}^{\infty} \sum_{l=1}^{d(k)} a_{k,l}(t, r) Y_{k,l}(\omega).$$

By orthogonality, we observe that $\|F(t, r\cdot)\|_{L^2_\omega} = \|a_{k,l}(t, r)\|_{l^2_{k,l}}$.

Due to the support assumptions for the Fourier inversion of f (denoted by \check{f}) we have that

$$(3.9) \quad \|f\|_{L^2(\mathbb{R}^n)}^2 \approx \int_0^\infty \int_{\mathbb{S}^{n-1}} |\check{f}(\rho\omega)|^2 d\omega d\rho.$$

If we expand the angular part of \check{f} using spherical harmonics, we find that if $\xi = \rho\omega$ with $\omega \in \mathbb{S}^{n-1}$, then there are generalized Fourier coefficients $c_k(\rho)$ which vanish when $\rho \notin [1/2, 1]$ so that

$$\check{f}(\xi) = \sum_{k,l} c_{k,l}(\rho) Y_{k,l}(\omega).$$

So, by (3.9) and Plancherel's theorem for \mathbb{S}^{n-1} and \mathbb{R} we have

$$(3.10) \quad \|f\|_{L^2(\mathbb{R}^n)}^2 \approx \sum_{k,l} \int_{\mathbb{R}} |c_{k,l}(\rho)|^2 d\rho \approx \sum_{k,l} \int_{\mathbb{R}} |\check{c}_{k,l}(s)|^2 ds,$$

if $\check{c}_{k,l}$ denotes the one-dimensional Fourier inversion of $c_{k,l}(\rho)$.

Recall that (see [32] Chapter IV Theorem 3.10 page 158)

$$(3.11) \quad c_{k,l}(\widehat{\rho Y_{k,l}(\omega)})(x) = g_{k,l}(|x|) Y_{k,l}\left(\frac{x}{|x|}\right),$$

where

$$(3.12) \quad g_{k,l}(r) = (2\pi)^{\frac{n}{2}} i^{-k} r^{-\frac{n-2}{2}} \int_0^\infty c_{k,l}(\rho) J_{k+\frac{n-2}{2}}(r\rho) \rho^{\frac{n}{2}} d\rho,$$

and J_m is the m -th Bessel function with $m \in \frac{1}{2}\mathbb{Z}$ and $m \geq 0$, we see that we have the formula (with $x = r\vartheta$ and $\vartheta \in \mathbb{S}^{n-1}$)

$$(3.13) \quad f(r\vartheta) = (2\pi)^{n/2} \sum_{k,l} i^{-k} r^{-\frac{n-2}{2}} \int_0^\infty c_{k,l}(\rho) J_{k+\frac{n-2}{2}}(r\rho) \rho^{\frac{n}{2}} d\rho Y_{k,l}(\vartheta).$$

We will use two integral representations of J_m as follows. The first is Schl\"afli's generalization of Bessel's integral (see Section 6.2 (4) page 176 in [40])

$$(3.14) \quad J_m(y) = \frac{e^{-im\pi/2}}{2\pi} \int_0^{2\pi} e^{iy \cos \theta - im\theta} d\theta - \frac{\sin(m\pi)}{\pi} \int_0^\infty e^{-y \sinh u - mu} d\theta.$$

The second is Lommel's expression of Bessel function (see Section 3.3 (1) page 47 in [40])

$$(3.15) \quad J_m(y) = \frac{(y/2)^m}{\sqrt{\pi} \Gamma(m+1/2)} \int_{-1}^1 e^{iyt} (1-t^2)^{m-\frac{1}{2}} dt.$$

Because of (3.13) and the support properties of the $c_{k,l}$, we find that if we fix $\eta \in C_0^\infty(\mathbb{R})$ satisfying $\eta(\tau) = 1$ for $1/2 \leq \tau \leq 1$ and $\eta(\tau) = 0$ for $\tau \notin [1/4, 2]$ and if we set $\alpha(\rho) = \rho^{n/2}\eta(\rho) \in \mathcal{S}(\mathbb{R})$, then we have

$$(e^{itD}f)(r\vartheta) = (2\pi)^{n/2} \sum_{k,l} i^{-k} r^{-\frac{n-2}{2}} I_{k,l}(t, r) Y_{k,l}(\vartheta),$$

where if we apply (3.14),

$$\begin{aligned} I_{k,l}(t, r) &= \int_0^\infty c_{k,l}(\rho) J_{k+\frac{n-2}{2}}(r\rho) e^{it\rho} \rho^{\frac{n}{2}} d\rho \\ &= \int_0^\infty \int_{-\infty}^\infty J_{k+\frac{n-2}{2}}(r\rho) e^{i\rho(t-s)} \check{c}_{k,l}(s) \alpha(\rho) ds d\rho \\ &= \frac{e^{-i(k+\frac{n-2}{2})\pi/2}}{2\pi} \int_0^\infty \int_{-\infty}^\infty e^{i\rho(t-s)} \check{c}_{k,l}(s) \alpha(\rho) \int_0^{2\pi} e^{i\rho r \cos \theta - i(k+\frac{n-2}{2})\theta} d\theta ds d\rho \\ &\quad - \frac{\sin((k+\frac{n-2}{2})\pi)}{\pi} \int_0^\infty \int_{-\infty}^\infty e^{i\rho(t-s)} \check{c}_{k,l}(s) \alpha(\rho) \int_0^\infty e^{-\rho r \sinh u - (k+(n-2)/2)u} du ds d\rho \\ &= e^{-i(k+\frac{n-2}{2})\pi/2} \int_{-\infty}^\infty \int_0^{2\pi} \check{c}_{k,l}(s) \check{\alpha}(t-s+r \cos \theta) e^{-i(k+\frac{n-2}{2})\theta} d\theta ds \\ &\quad - 2 \sin((k+\frac{n-2}{2})\pi) \int_{-\infty}^\infty \int_0^\infty \check{c}_{k,l}(s) e^{-(k+(n-2)/2)u} \check{\beta}(t-s, r, u) du ds, \end{aligned}$$

where $\beta(\rho, r, u) = \alpha(\rho) e^{-\rho r \sinh u}$ and the inverse Fourier transformation acts on the first variable of β . And if we apply (3.15) instead of (3.14), then

$$\begin{aligned} I_{k,l}(t, r) &= \frac{1}{2^{k+\frac{n-2}{2}} \sqrt{\pi} \Gamma(k+\frac{n-1}{2})} \\ &\quad \times \int_0^\infty \int_{-\infty}^\infty e^{i\rho(t-s)} \check{c}_{k,l}(s) \alpha(\rho) (\rho r)^{k+\frac{n-2}{2}} \int_{-1}^1 e^{i\rho r u} (1-u^2)^{k+\frac{n-3}{2}} du ds d\rho \\ &= 2\pi \frac{r^{k+\frac{n-2}{2}}}{2^{k+\frac{n-2}{2}} \sqrt{\pi} \Gamma(k+\frac{n-1}{2})} \int_{-\infty}^\infty \int_{-1}^1 \check{c}_{k,l}(s) \check{\gamma}(t-s+ru) (1-u^2)^{k+\frac{n-3}{2}} du ds, \end{aligned}$$

where $\gamma(\rho) = \alpha(\rho) \rho^{k+\frac{n-2}{2}}$. For simplicity, we introduce new functions $\psi_{ik}(m, r)$ ($i = 1, 2, 3$)

$$(3.16) \quad \psi_{1k}(m, r) = \int_0^{2\pi} e^{-i(k+\frac{n-2}{2})\theta} \check{\alpha}(m+r \cos \theta) d\theta,$$

$$(3.17) \quad \psi_{2k}(m, r) = \sin((k+\frac{n-2}{2})\pi) \int_0^\infty e^{-(k+\frac{n-2}{2})u} \check{\beta}(m, r, u) du,$$

$$(3.18) \quad \psi_{3k}(m, r) = \frac{2\pi}{2^{k+\frac{n-2}{2}} \sqrt{\pi} \Gamma(k+\frac{n-1}{2})} r^{k+\frac{n-2}{2}} \int_{-1}^1 \check{\gamma}(m+ru) (1-u^2)^{k+\frac{n-3}{2}} du.$$

Thus

$$\begin{aligned} I_{k,l}(t, r) &= e^{-i(k+\frac{n-2}{2})\pi/2} \int_{\mathbb{R}} \check{c}_{k,l}(s) \psi_{1k}(t-s, r) ds - 2 \int_{\mathbb{R}} \check{c}_{k,l}(s) \psi_{2k}(t-s, r) ds \\ &= \int_{\mathbb{R}} \check{c}_{k,l}(s) \psi_{3k}(t-s, r) ds . \end{aligned}$$

As a result, we have that for any $r > 0$,

$$(3.19) \quad \|(e^{itD} f)(r\vartheta)\|_{L_{\vartheta}^2} \lesssim \sum_{i=1,2} \left\| \int_{\mathbb{R}} \check{c}_{k,l}(s) \psi_{ik}(t-s, r) r^{-\frac{n-2}{2}} ds \right\|_{l_{k,l}^2},$$

and

$$(3.20) \quad \|(e^{itD} f)(r\vartheta)\|_{L_{\vartheta}^2} \lesssim \left\| \int_{\mathbb{R}} \check{c}_{k,l}(s) \psi_{3k}(t-s, r) r^{-\frac{n-2}{2}} ds \right\|_{l_{k,l}^2}.$$

Now we claim that we have the following estimates

$$(3.21) \quad \|\psi_{ik}(m, r) \langle m \rangle^{\frac{n-1}{2}} r^{-\frac{n-2}{2}}\|_{L_m^2} \leq C, \text{ for } i = 1, 2 \text{ and } r > 1,$$

$$(3.22) \quad \|\psi_{1k}(m, r) \langle m \rangle^{\frac{n-1}{2}} r^{-\frac{n-2}{2}}\|_{L_m^2} \leq C, \text{ for } n \text{ even and } r \leq 1,$$

$$(3.23) \quad \|\psi_{3k}(m, r) \langle m \rangle^{\frac{n-1}{2}} r^{-\frac{n-2}{2}}\|_{L_m^2} \leq C, \text{ for } n \text{ odd and } r \leq 1,$$

where $\langle m \rangle = \sqrt{1+m^2}$ and C is independent of $k \in \mathbb{Z}$ and $r > 0$. Based on these estimates, we have

$$\begin{aligned} &\|(e^{itD} f)(r\vartheta)\|_{L_{\vartheta}^2} \\ &\lesssim \sum_{i=1}^2 \|\check{c}_{k,l}(s) \psi_{ik}(t-s, r) r^{-\frac{n-2}{2}}\|_{l_{k,l}^2 L_s^1} \\ &\lesssim \sum_{i=1}^2 \|\check{c}_{k,l}(s) \langle t-s \rangle^{-(n-1)/2}\|_{l_{k,l}^2 L_s^2} \|\langle t-s \rangle^{\frac{n-1}{2}} r^{-\frac{n-2}{2}} \psi_{ik}(t-s, r)\|_{L_s^2} \\ &\lesssim \|\check{c}_{k,l}(s) \langle t-s \rangle^{-(n-1)/2}\|_{l_{k,l}^2 L_s^2} \end{aligned}$$

for $r > 1$. For the estimates with $r \leq 1$, we need only to use the same argument with the observation that $\psi_{2k} = 0$ when n is even. In summary, these estimates tell us that

$$\|(e^{itD} f)(x)\|_{L_{|x|}^{\infty} L_{\omega}^2} \leq C \|\check{c}_{k,l}(s) \langle t-s \rangle^{-(n-1)/2}\|_{l_{k,l}^2 L_s^2}.$$

Recall that the energy estimates and (3.10) tell us that

$$\|(e^{itD} f)(x)\|_{L_{|x|}^2 L_{\omega}^2} \leq \|f\|_{L^2} \leq C \|\check{c}_{k,l}(s)\|_{l_{k,l}^2 L_s^2}.$$

By interpolation, we can immediately get

$$(3.24) \quad \|(e^{itD} f)(x)\|_{L_{|x|}^r L_{\omega}^2} \leq C \|\check{c}_{k,l}(s) \langle t-s \rangle^{-(n-1)(1/2-1/r)}\|_{l_{k,l}^2 L_s^2}.$$

Now we see that we have the estimates (3.1), for any $q \geq 2$ and $1/q < (n-1)(1/2 - 1/r)$ or $(q, r) = (\infty, 2)$. In fact,

$$\begin{aligned} \|(e^{itD}f)(x)\|_{L_t^q L_{|x|}^r L_\omega^2} &\leq C \|\check{c}_{k,l}(s) \langle t-s \rangle^{-(n-1)(1/2-1/r)}\|_{L_t^q L_{k,l}^2 L_s^2} \\ &\leq C \|\check{c}_{k,l}(s) \langle t-s \rangle^{-(n-1)(1/2-1/r)}\|_{L_{k,l}^2 L_s^2 L_t^q} \\ &\leq C \|\check{c}_{k,l}(s)\|_{L_{k,l}^2 L_s^2} \simeq \|f\|_{L^2}. \end{aligned}$$

Similarly, we can also prove (3.2) for $1/q \geq (n-1)(1/2 - 1/r)$.

3.2. The estimates for $\psi_{ik}(m, r)$. Now we present the proof of the key estimates (3.21)-(3.23) for $\psi_{ik}(m, r)$, to conclude the proof of (3.1).

At first, we observe that the estimate (3.21) for ψ_{1k} has been obtained in [27] and [6]. Moreover, $\psi_{2k} = 0$ when n is even. Thus we need only to prove (3.21) for ψ_{2k} with $n \geq 3$ odd, (3.22) for ψ_{1k} and (3.23) for ψ_{3k} .

Lemma 3.1. *Let $n \geq 3$ be odd, $\beta(\rho, r, u) = \alpha(\rho)e^{-\rho r \sinh u}$ and $r > 1$. Then there is a uniform constant C , which is independent of k and $r > 1$ so that the following inequality hold*

$$(3.25) \quad \|\psi_{2k}(m, r) \langle m \rangle^{\frac{n-1}{2}} r^{-\frac{n-2}{2}}\|_{L_m^2} \leq C.$$

Proof. Notice that $\beta \in \mathcal{S}(\mathbb{R})$ with respect to all variables (together with the support in $[1/4, 2]$ for ρ). If we use Hölder's inequality, Plancherel Theorem and the facts $r > 1$ and $k + \frac{n-2}{2} \geq \frac{1}{2}$, then

$$\begin{aligned} \|\psi_{2k}(m, r) \langle m \rangle^{\frac{n-1}{2}} r^{-\frac{n-2}{2}}\|_{L_m^2} &= \|\langle m \rangle^{\frac{n-1}{2}} r^{-\frac{n-2}{2}} \int_0^\infty \check{\beta}(m, r, u) e^{-(k+\frac{n-2}{2})u} du\|_{L_m^2} \\ &\lesssim \|\check{\beta}(m, r, u) \langle m \rangle^{\frac{n-1}{2}}\|_{L_u^2 L_m^2} \\ &\lesssim \|\beta(\rho, r, u)\|_{L_u^2 H_\rho^{\frac{n-1}{2}}} \\ &\lesssim \sum_{0 \leq l \leq \frac{n-1}{2}} \|(r \sinh u)^{\frac{n-1}{2}-l} e^{-\rho r \sinh u} \alpha^{(l)}(\rho)\|_{L_\rho^2 L_u^2} \\ &\lesssim \|\langle r \sinh u \rangle^{\frac{n-1}{2}} e^{-\frac{r \sinh u}{2}}\|_{L_u^2} \\ &\leq C. \end{aligned}$$

□

Lemma 3.2. *If $n \geq 2$ is even, then for any $r \in (0, 1]$ and $N > 0$, we have*

$$(3.26) \quad |\psi_{1k}(m, r)| \leq C_N r^{(n-2)/2} \langle m \rangle^{-N}.$$

Consequently, we have (3.22) for ψ_{1k} .

Proof. First, observe that the case $n = 2$ is trivial since $\alpha \in \mathcal{S}$. For the case $n \geq 4$ and even, then $\frac{n-2}{2} \in \mathbb{N}$. The estimate (3.26) follows immediately, if we use the Taylor expansion of $\check{\alpha}$ up to order $n/2$, in terms of $r \cos \theta$, and recall that we have the orthogonality relation

$$\int_0^{2\pi} e^{i(k+\frac{n-2}{2})\theta} (\cos \theta)^j d\theta = 0, \quad \text{if } 0 \leq j < \frac{n-2}{2} \Leftrightarrow 0 \leq j \leq \frac{n-4}{2}.$$

■

Lemma 3.3. *If $n \geq 3$ is odd, the estimate (3.23) holds for ψ_{3k} .*

Proof. Notice that $r < 1$ and $\gamma(\rho) = \alpha(\rho)\rho^{k+\frac{n-2}{2}} = \eta(\rho)\rho^{k+n-1}$,

$$\begin{aligned}
& \|\psi_{3k}(m, r) \langle m \rangle^{\frac{n-1}{2}} r^{-\frac{n-2}{2}}\|_{L_m^2} \\
&= \left\| \frac{2\pi}{2^{k+\frac{n-2}{2}} \sqrt{\pi} \Gamma(k + \frac{n-1}{2})} \langle m \rangle^{\frac{n-1}{2}} r^k \int_{-1}^1 \tilde{\gamma}(m+ru)(1-u^2)^{k+\frac{n-3}{2}} du \right\|_{L_m^2} \\
&\lesssim \frac{1}{2^k \Gamma(k + \frac{n-1}{2})} \int_{-1}^1 \|\langle m \rangle^{\frac{n-1}{2}} \tilde{\gamma}(m+ru)\|_{L_m^2} (1-u^2)^{k+\frac{n-3}{2}} du \\
&\leq \frac{1}{2^k \Gamma(k + \frac{n-1}{2})} \int_{-1}^1 \|\gamma(\rho)\|_{H_\rho^{\frac{n-1}{2}}} (1-u^2)^{k+\frac{n-3}{2}} du \\
&\lesssim \frac{1}{2^k \Gamma(k + \frac{n-1}{2})} \frac{(k+n-1)!}{(k + \frac{n-1}{2})!} \\
&= \frac{(k+n-1)(k+n-2) \cdots (k + \frac{n+1}{2})}{2^k (k + \frac{n-3}{2})!}
\end{aligned}$$

When $k \geq \frac{n+1}{2}$, we have $k+n-j \leq 2k+n-1-2j$ for $1 \leq j \leq \frac{n-1}{2}$, and we see that the last quantity is less than or equal to 1. For any $k < \frac{n+1}{2}$, the last quantity is bounded, and this concludes the proof of (3.23) with a constant independent of k . ■

4. RADIAL WEIGHTED STRICHARTZ ESTIMATES, A MOTIVATION

In this section we study the weighted Strichartz estimates for the wave equation with radial initial data and prove Theorem 1.5. The argument of [33] in proving Strichartz estimates of the wave equation with radial initial data can be adapted for our purpose.

Proof of Theorem 1.5 Let f_1 be a radially symmetric unit frequency function and $u_1 = e^{itD} f_1$. For a radially symmetric initial data f_1 , we can write u_1 as the integral formula:

$$(4.1) \quad e^{itD} f_1(r) = Cr^{-\frac{n-2}{2}} \int_0^\infty e^{it\rho} J_{\frac{n-2}{2}}(r\rho) \widehat{f_1}(\rho) \rho^{\frac{n}{2}} d\rho,$$

where $J_{\frac{n-2}{2}}(y)$ is the Bessel function of order $\frac{n-2}{2}$ (compare (3.13)). Then we use the well known asymptotics for Bessel functions of relatively small order (see [40]):

$$(4.2) \quad J_{\frac{n-2}{2}}(y) = \begin{cases} \left(\frac{2}{\pi y}\right)^{\frac{1}{2}} \left[\cos(y - \frac{n-1}{4}\pi) \cdot m_1(y) - \sin(y - \frac{n-1}{4}\pi) \cdot m_2(y) \right], & \text{for } y \geq 1, \\ y^{\frac{n-2}{2}} \cdot m_3(y), & \text{for } 0 \leq y \leq 1. \end{cases}$$

Here the function m_3 is smooth, and the remaining m_i have asymptotic expansions:

$$\begin{aligned} m_1(y) &= \sum_k C_{1,k} y^{-2k}, \\ m_2(y) &= \sum_k C_{2,k} y^{-2k-1}, \end{aligned}$$

as $y \rightarrow \infty$. In other words, the functions $m_1(r\rho)$ and $m_2(r\rho)$ are smooth with derivatives in ρ uniformly bounded for all $\frac{1}{2} \leq \rho \leq 2$ and $r \geq 2$. Substituting the asymptotic formula (4.2) into the integral formula, we may assume without loss of generality that we are trying to bound integrals of the form:

$$(4.3) \quad I^\pm(t, r) = \frac{1}{(1+r)^{\frac{n-1}{2}}} \int_{-\infty}^{\infty} e^{i(t \pm r)\rho} m^\pm(r, \rho) \chi_{(1/4, 4)}(\rho) \hat{f}_1(\rho) d\rho,$$

where m^\pm is a smooth function with derivatives in ρ uniformly bounded for all $r \geq 0$, and $\chi_{(1/4, 4)}$ is a smooth bump function on the interval $(\frac{1}{4}, 4)$. It is now apparent that the integrals in (4.3) are essentially time translated inverse Fourier transforms of a one dimensional unit frequency function. Therefore, we can localize these integrals in physical space (i.e. the $t \pm r$ variable) on a $O(1)$ scale. Since the function \hat{f}_1 is compactly supported in the interval $(0, 4)$, we may take its Fourier series development:

$$(4.4) \quad \hat{f}_1(\rho) = \sum_{k=-\infty}^{\infty} c_k e^{ik\rho}, \quad \rho \in (0, 4).$$

An important thing to notice here is that we can recover the L^2 norm of f_1 as a function on \mathbb{R}^n in terms of $\{c_k\}$:

$$(4.5) \quad \|f_1\|_{L_x^2}^2 \sim \|\hat{f}_1\|_{L_\rho^2}^2 \sim \sum_k |c_k|^2.$$

Sticking the series (4.4) into the the integrals (4.3) yields:

$$(4.6) \quad I^\pm(t, r) = \sum_k \frac{c_k}{(1+r)^{\frac{n-1}{2}}} \psi_k^\pm(t, r),$$

where

$$\psi_k^\pm(t, r) = \int_{-\infty}^{\infty} e^{i(t \pm r + k)\rho} m^\pm(r, \rho) \chi_{(1/4, 4)}(\rho) d\rho.$$

Integrating by parts as many times as necessary in the above formula, we see that we have the asymptotic bound:

$$(4.7) \quad |\psi_k^\pm(t, r)| \leq \frac{C_M}{(1 + |t \pm r + k|)^{2M}}, \quad \forall M \in \mathbb{N}.$$

Using the expansion (4.6) and the asymptotic bound (4.7) we can directly compute that

$$\begin{aligned} \| |x|^{-\alpha} I^\pm(t, |x|) \|_{L_x^p} &\lesssim \| c_k \psi_k^\pm(t, r) (1+r)^{-\frac{n-1}{2}} r^{\frac{n-1}{p}-\alpha} \|_{L_r^p \ell_k^1} \\ &\lesssim \| c_k (1 + |t \pm r + k|)^{-2M} (1+r)^{-\frac{n-1}{2}} r^{\frac{n-1}{p}-\alpha} \|_{L_r^p \ell_k^1} \\ &\lesssim \| c_k (1 + ||t + k| - r|)^{-M} (1+r)^{-\frac{n-1}{2}} r^{\frac{n-1}{p}-\alpha} \|_{L_r^p \ell_k^p}. \end{aligned}$$

The manipulation to get the last line above follows from Hölder's inequality. Note that to make the function integrable in L^p , we must have α be the number such that $\frac{n-1}{p} - \alpha > -\frac{1}{p}$ (or $-\alpha \geq 0$ when $p = \infty$), i.e.,

$$(4.8) \quad \alpha < \frac{n}{p} \quad (\alpha \leq 0 \text{ if } p = \infty).$$

If we also choose M large enough, then by integrating each expression in this line term by term, we arrive at the bound:

$$\| |x|^{-\alpha} I^\pm(t, |x|) \|_{L_x^p} \lesssim \| c_k (1 + |t + k|)^{-(n-1)(\frac{1}{2} - \frac{1}{p}) - \alpha} \|_{\ell_k^p}.$$

Testing this last expression for L^q in time, and using the inclusion $\ell^{\min\{p,q\}} \subseteq \ell^p$, we see that:

$$\begin{aligned} \| |x|^{-\alpha} I^\pm(t, |x|) \|_{L_t^q L_x^p} &\lesssim \| c_k (1 + |t + k|)^{-(n-1)(\frac{1}{2} - \frac{1}{p}) - \alpha} \|_{L_t^q \ell_k^p} \\ &\lesssim \| c_k (1 + |t + k|)^{-(n-1)(\frac{1}{2} - \frac{1}{p}) - \alpha} \|_{\ell_k^{\min\{p,q\}} L_t^q} \\ &\lesssim \| c_k \|_{\ell_k^2}, \end{aligned}$$

as long as $\min\{p, q\} \geq 2$ and $\frac{1}{q} < (n-1)(\frac{1}{2} - \frac{1}{p}) + \alpha$ (or $0 \leq (n-1)(\frac{1}{2} - \frac{1}{p}) + \alpha$ for $q = \infty$).

In conclusion, using the characterization (4.5), we know that if we have $q, p \geq 2$ and

$$(4.9) \quad \frac{1}{q} - (n-1)(\frac{1}{2} - \frac{1}{p}) < \alpha < \frac{n}{p}$$

(we can take the first inequality with equality when $q = \infty$ and the second inequality with equality when $p = \infty$), then we have

$$(4.10) \quad \| |x|^{-\alpha} e^{itD} f_1 \|_{L_t^q L_x^p} \lesssim \| f_1 \|_{L^2}.$$

Now we use the Littlewood-Paley decomposition $f = \sum_{j \in \mathbb{Z}} f_j$, and apply (4.10) to get

$$\begin{aligned} \| |x|^{-\alpha} e^{itD} f \|_{L_t^q L_x^r} &= \| |x|^{-\alpha} e^{itD} \sum_{j \in \mathbb{Z}} f_j \|_{L_t^q L_x^r} \\ &\lesssim \sum_{j \in \mathbb{Z}} \| |x|^{-\alpha} e^{itD} f_j \|_{L_t^q L_x^r} \\ &\lesssim \sum_{j \in \mathbb{Z}} (2^{js} \| f_j \|_{L_x^2}) \quad \text{where } s = \frac{n}{2} + \alpha - \frac{1}{q} - \frac{n}{r} \\ &\lesssim \| 2^{js} f_j \|_{l_j^1 L_x^2} \\ &\simeq \| f \|_{\dot{B}_{2,1}^s}. \end{aligned}$$

Next we use real interpolation to prove the estimate (1.18).

Proof of Theorem 1.5: The case $(q, r) = (2, 2)$ follows directly from the weighted Strichartz estimates (1.22) with $r = 2$ and $a = 1$. If $q = \infty$ and $2 \leq r < \infty$, it follows directly from the weighted Hardy-Littlewood-Sobolev inequality of Stein and Weiss [31]

$$(4.11) \quad \| |x|^{-\alpha} f \|_{L^r(\mathbb{R}^n)} \lesssim \| |x|^\beta D^s f \|_{L^q(\mathbb{R}^n)}$$

with $1 < q \leq r < \infty$, $\alpha < n/r$, $\beta < n/q'$, $\alpha + \beta \geq 0$, $s \in (0, n)$ and $-s + n/q + \beta = -\alpha + n/r$.

For fixed $2 < r < \infty$ and α , we can choose $2 < r_1 < r < r_2 < \infty$ such that

$$\frac{2}{r} = \frac{1}{r_1} + \frac{1}{r_2}, \quad \frac{1}{q} - (n-1)\left(\frac{1}{2} - \frac{1}{r_i}\right) < \alpha < \frac{n}{r_i}.$$

Let $d\omega_i = |x|^{-\alpha r_i} dx$, $i = 1, 2$, from the discussion above we have

$$(4.12) \quad \|e^{itD} f\|_{L_t^2 L_x^{r_1}(d\omega_1)} \lesssim \|f\|_{\dot{B}_{2,1}^{s_1}}$$

$$(4.13) \quad \|e^{itD} f\|_{L_t^2 L_x^{r_2}(d\omega_2)} \lesssim \|f\|_{\dot{B}_{2,1}^{s_2}}$$

where $s_i = \frac{n}{2} + \alpha - \frac{1}{q} - \frac{n}{r_i}$, $i = 1, 2$.

Since $(L_x^{r_1}(d\omega_1), L_x^{r_2}(d\omega_2))$ is an interpolation couple, by Theorem 1.18.4 in [39] we have

$$(L_t^q(L_x^{r_1}(d\omega_1)), L_t^q(L_x^{r_2}(d\omega_2)))_{\theta, 2} = L_t^q((L_x^{r_1}(d\omega_1), L_x^{r_2}(d\omega_2))_{\theta, 2}).$$

And from Theorem 6.4.5 in [1] we know:

$$(B_{p,q_0}^{s_1}, B_{p,q_1}^{s_2})_{\theta, r} = B_{p,r}^s, \text{ if } s_0 \neq s_1, 0 < \theta < 1, r, q_0, q_1 \geq 1, s = (1-\theta)s_1 + \theta s_2.$$

Now by real interpolation between (4.12) and (4.13), we get

$$\begin{aligned} \| |x|^{-\alpha} e^{itD} f \|_{L_t^q L_x^r} &\lesssim \| e^{itD} f \|_{L_t^q((L_x^{r_1}(d\omega_1), L_x^{r_2}(d\omega_2))_{1/2, r})} \\ &\lesssim \| e^{itD} f \|_{(L_t^q L_x^{r_1}(d\omega_1), L_t^q L_x^{r_2}(d\omega_2))_{1/2, r}} \\ &\lesssim \| f \|_{(\dot{B}_{2,1}^{s_1}, \dot{B}_{2,1}^{s_2})_{1/2, r}} \\ &= \| f \|_{\dot{B}_{2,r}^s} \\ &\lesssim \| f \|_{\dot{H}^s}. \end{aligned}$$

This proves (1.18) for $2 < r < \infty$, $2 \leq q \leq \infty$.

Likewise, we can prove the case when $2 < q < \infty$, $2 \leq r \leq \infty$. ■

5. WEIGHTED STRICHARTZ ESTIMATES

In this section, we prove the weighted Strichartz estimates stated in Theorem 1.6, based on Rodnianski's argument, the weighted Strichartz estimates (1.22) and a localized version of the weighted HLS estimates.

5.1. Localized Weighted Hardy-Littlewood-Sobolev Inequality. Recall that we have the classical weighted Hardy-Littlewood-Sobolev inequality (4.11) of Stein and Weiss [31], i.e.,

$$\| |x|^{-\alpha} f \|_{L^r(\mathbb{R}^n)} \lesssim \| |x|^\beta \mathcal{D}^s f \|_{L^q(\mathbb{R}^n)}$$

with $1 < q \leq r < \infty$, $\alpha < n/r$, $\beta < n/q'$, $\alpha + \beta \geq 0$, $s \in (0, n)$ and $-s + n/q + \beta = -\alpha + n/r$.

In particular, if we choose $\beta = -\alpha$, the corresponding estimate on \mathbb{R}^n is true for $-n/q' < \alpha < n/r$, $1 < q \leq r < \infty$ and $s = n/q - n/r$.

In this subsection, we are interested in the localized version of the estimate with $\beta = -\alpha$. More precisely, if we denote B_1 be the unit ball, and B_2 the ball centered at origin with radius 2, then we aim at the proof of the following lemma.

Lemma 5.1 (Localized Weighted HLS). *Let $1 < q \leq r < \infty$, and $-n/q' < \alpha < n/r$, we have the localized version of the weighted Hardy-Littlewood-Sobolev inequality*

$$(5.1) \quad \| |x|^{-\alpha} f \|_{L^r_{x \in B_1}} \lesssim \sum_{|\beta| \leq s} \| |x|^{-\alpha} \partial^\beta f \|_{L^q_{B_2}}$$

if s is large enough (in fact, we need only to choose $s = 2m$ with $m > n/2$).

Proof. We will prove the result for $s = 2m$ with $n/2 < m \in \mathbb{N}$. At first, we observe that the proof of the estimate (5.1) can be reduced to the proof of the following

$$(5.2) \quad \| |x|^{-\alpha} \phi f \|_{L^r} \lesssim \| |x|^{-\alpha} \Lambda^{2m} f \|_{L^q},$$

where $\phi \in C_0^\infty$ and $\Lambda^{2m} = (1 - \Delta)^m$ is a differential operator. In fact, if this estimate is true for any f , we can choose $\phi = 1$ in B_1 , $\phi \psi = \phi$, $\psi = 0$ in B_2^c , and $f = \psi g$. Recall that

$$[\Lambda^{2m}, \psi]g = \sum_{|\alpha|=1}^{2m} \sum_{|\alpha|+|\beta| \leq 2m} c_{\alpha,\beta} \partial^\alpha \psi \partial^\beta g,$$

and $\text{supp } \partial \psi \subset B_2 \setminus B_1$. We have

$$\begin{aligned} \| |x|^{-\alpha} g \|_{L^r_{x \in B_1}} &\leq \| |x|^{-\alpha} \phi \psi g \|_{L^r} \\ &\lesssim \| |x|^{-\alpha} \Lambda^{2m}(\psi g) \|_{L^q} \\ &\lesssim \| |x|^{-\alpha} \psi \Lambda^{2m} g \|_{L^q} + \sum_{|\beta|=0}^{2m-1} \| \partial^\beta g \|_{L^q_{B_2 \setminus B_1}} \\ &\lesssim \| |x|^{-\alpha} \Lambda^{2m} g \|_{L^q_{B_2}} + \sum_{|\beta|=0}^{2m-1} \| \partial^\beta g \|_{L^q_{B_2 \setminus B_1}} \\ &\lesssim \sum_{|\beta|=0}^{2m} \| |x|^{-\alpha} \partial^\beta g \|_{L^q_{B_2}}. \end{aligned}$$

So it is sufficient to prove (5.2). First we write (5.2) in the equivalent form

$$(5.3) \quad \| |x|^{-\alpha} \phi \Lambda^{-2m} |x|^\alpha f \|_{L^r} \lesssim \| f \|_{L^q}.$$

Recall that $\Lambda^{-2m} f = \mathcal{F}^{-1}(1 + |\xi|^2)^{-m} \hat{f} = K * f$, and if $m > n/2$, then $K(x) = O(\langle x \rangle^{-N})$ for any N . By introducing $\psi \in C_0^\infty$ such that $\psi \phi = \phi$, if $\alpha < n/r$, we can control

$$\| |x|^{-\alpha} \phi \Lambda^{-2m} |x|^\alpha f \|_{L^r} \lesssim \| |x|^{-\alpha} \psi \|_{L^r} \| \phi \Lambda^{-2m} |x|^\alpha f \|_{L^\infty} \lesssim \| T f \|_{L^\infty}$$

with

$$T f(x) = \int \phi(x) K(x-y) |y|^\alpha f(y) dy.$$

By Hölder's inequality, we can estimate $\|Tf\|_{L^\infty}$ by $\|f\|_{L^q}$, if we have

$$\phi(x)K(x-y)|y|^\alpha \in L_x^\infty L_y^{q'},$$

which can be easily seen to hold if we have $\alpha > -n/q'$. ■

5.2. Rodnianski's Argument for the Weighted Estimates. Let $f_{1,N}$ be a unit frequency function of angular frequency N and $u_{1,N} = e^{itD^a} f_{1,N}$ with $a > 0$.

First, using the Sobolev inequality $\Lambda_\omega^{-\frac{n-1}{2}} L_\omega^2 \subset L_\omega^\infty$ on the unit sphere \mathbb{S}^{n-1} for angular frequency localized functions (see (7.3) in Appendix 7), we have the following estimate for any tiling of \mathbb{R}^n by cubes $\{Q_\alpha\}$:

$$\begin{aligned} \| |x|^{-\alpha} u(t) \|_{L^2(Q_\alpha)} &\lesssim (\|1\|_{L_\omega^2(|x|\omega \in Q_\beta)} \| |x|^{-\alpha} u(t) \|_{L_\omega^\infty})_{L_{|x|}^2} \\ &\lesssim \| |x|^{-\frac{n-1}{2}} |x|^{-\alpha} \Lambda_\omega^{\frac{n-1}{2}} u(t) \|_{L_x^2}, \end{aligned}$$

where we have used the fact that

$$|\{\omega : |x|\omega \in Q_\alpha\}| \lesssim |x|^{-(n-1)}.$$

This means that we have

$$(5.4) \quad \| |x|^{-\alpha} u(t) \|_{\ell^\infty L_Q^2} \lesssim \| |x|^{-\frac{n-1}{2}-\alpha} \Lambda_\omega^{\frac{n-1}{2}} u(t) \|_{L_x^2}.$$

Interpolating this with the trivial estimate

$$\| |x|^{-\alpha} u(t) \|_{\ell^2 L_Q^2} \lesssim \| |x|^{-\alpha} u(t) \|_{L_x^2},$$

we arrive at the following estimate ($2 \leq r$):

$$(5.5) \quad \| |x|^{-\alpha} u \|_{L_t^2 \ell^r L_Q^2} \lesssim \| |x|^{-d-\alpha} \Lambda_\omega^d u \|_{L_t^2 L_x^2},$$

where $d = (n-1)(\frac{1}{2} - \frac{1}{r})$.

Recall that we have the generalized Morawetz estimates (1.23), i.e.,

$$(5.6) \quad \| |x|^{-\frac{b}{2}} e^{itD^a} f(x) \|_{L_{t,x}^2} \lesssim \| D^{\frac{b-a}{2}} \Lambda_\omega^{\frac{1-b}{2}} f \|_{L_x^2},$$

if $b \in (1, n)$, $a > 0$. Combining (5.5) and (5.6), we have

$$(5.7) \quad \| |x|^{-\alpha} e^{itD^a} f_{1,N} \|_{L_t^2 \ell^r L_Q^2} \lesssim \| \Lambda_\omega^{\frac{1}{2}-\alpha} f_{1,N} \|_{L_x^2},$$

if $\frac{2-n}{2} + \frac{n-1}{r} < \alpha < \frac{1}{2} + \frac{n-1}{r}$ and $2 \leq r$.

We can see that (5.7) implies the nonhomogeneous weighted Strichartz estimates, i.e., we have

$$(5.8) \quad \| \langle x \rangle^{-\alpha} e^{itD^a} f_1 \|_{L_t^2 L_x^r} \lesssim \| \Lambda_\omega^{\frac{1}{2}-\alpha} f_1 \|_{L_x^2},$$

if $\frac{2-n}{2} + \frac{n-1}{r} < \alpha < \frac{1}{2} + \frac{n-1}{r}$ and $2 \leq r < \infty$.

To see this, we can compute, using the Sobolev embedding on Q_β , that for any $r \geq 2$,

$$\begin{aligned}
\|\langle x \rangle^{-\alpha} e^{itD^a} f_{1,N}\|_{L_t^2 L_x^r} &\simeq \|\langle r_\beta \rangle^{-\alpha} e^{itD^a} f_{1,N}\|_{L_t^2 \ell_\beta^r L^r(Q_\beta)} \\
&\lesssim \|\langle r_\beta \rangle^{-\alpha} e^{itD^a} (1 - \Delta)^n f_{1,N}\|_{L_t^2 \ell_\beta^r L^2(Q_\beta)} \\
&\lesssim \| |x|^{-\alpha} e^{itD^a} (1 - \Delta)^n f_{1,N}\|_{L_t^2 \ell^r L_Q^2} \\
&\lesssim \|\Lambda_\omega^{1/2-\alpha} (1 - \Delta)^n f_{1,N}\|_{L_x^2} \\
&\lesssim \|\Lambda_\omega^{1/2-\alpha} f_{1,N}\|_{L_x^2}.
\end{aligned}$$

Then an application of Littlewood-Paley-Stein inequality gives us (5.8) for $r < \infty$.

This gives the proof of (1.20) in the region $|x| \geq 1$, for frequency localized functions.

To prove the homogeneous estimates for $|x| \leq 1$, we use the localized weighted HLS estimates proven in Lemma 5.1. The estimate is

$$(5.9) \quad \| |x|^{-\alpha} f_{1,N} \|_{L_{x \in B_1}^r} \lesssim \sum_{|\beta| \leq s} \| |x|^{-\alpha} \partial^\beta f_{1,N} \|_{L_{B_2}^q},$$

if $1 < q \leq r < \infty$, $-n/q' < \alpha < n/r$ and s is large enough. Then by (5.7) and (5.9) with $q = 2$, we have

$$\begin{aligned}
\| |x|^{-\alpha} e^{itD^a} f_{1,N} \|_{L_t^2 L_{B_1}^r} &\lesssim \sum_{|\beta| \leq s} \| |x|^{-\alpha} e^{itD^a} \partial^\beta f_{1,N} \|_{L_t^2 L_{B_2}^2} \\
&\lesssim \sum_{|\beta| \leq s} \| |x|^{-\alpha} e^{itD^a} \partial^\beta f_{1,N} \|_{L_t^2 \ell^r L_Q^2} \\
&\lesssim \sum_{|\beta| \leq s} \|\Lambda_\omega^{\frac{1}{2}-\alpha} \partial^\beta f_{1,N}\|_{L_x^2}.
\end{aligned}$$

Recall that $[\Omega_{ij}, \partial_k] = \delta_{jk} \partial_i - \delta_{ik} \partial_j$ and $f_1 = \phi * f$ for some spectral localized radial bump function ϕ , we have the following estimates for the even positive numbers s

$$\|\Lambda_\omega^s \partial^\beta \phi * f\|_{L_x^2} \lesssim \|\Lambda_\omega^s f\|_{L_x^2}.$$

By duality and interpolation, the same estimates are true for a general real number s . Applying this estimate in the previous inequalities, we find that

$$(5.10) \quad \| |x|^{-\alpha} e^{itD^a} f_{1,N} \|_{L_t^2 L_{B_1}^r} \lesssim \|\Lambda_\omega^{\frac{1}{2}-\alpha} f_{1,N}\|_{L_x^2}$$

if $2 \leq r < \infty$, $-n/2 < \alpha < n/r$.

Combining (5.8) and (5.10), we get

$$(5.11) \quad \| |x|^{-\alpha} e^{itD^a} f_{1,N} \|_{L_t^2 L_x^r} \lesssim \|\Lambda_\omega^{\frac{1}{2}-\alpha} f_{1,N}\|_{L_x^2}$$

for

$$\frac{2-n}{2} + \frac{n-1}{r} < \alpha < \frac{n}{r}, \quad 2 \leq r < \infty.$$

An application of Littlewood-Paley-Stein inequality gives us (5.11) for f_1 since $2 \leq r < \infty$.

Now we use the Littlewood-Paley decomposition $f = \sum_{j \in \mathbb{Z}} f_j$, and apply (5.11) to get

$$\| |x|^{-\alpha} e^{itD^a} f \|_{L_t^2 L_x^r} \simeq \| \Lambda_\omega^{\frac{1}{2}-\alpha} f \|_{\dot{B}_{2,1}^s}.$$

for $s = \frac{n-a}{2} + \alpha - \frac{n}{r}$

Next we can use real interpolation as in section 4 to get the case $q = 2 < r < \infty$ proved. Recall that we have the weighted Strichartz estimates (1.22) for $2 \leq q = r \leq \infty$, and hence (1.20) with $2 \leq q = r < \infty$ (see e.g. (2.4)), we have the estimate (1.20) for $2 \leq q \leq r < \infty$, by using real interpolation again.

5.3. Weighted Estimates for $q \geq r$. In this subsection we will prove Theorem 1.7.

Recall that Hardy's inequality gives,

$$(5.12) \quad \| |x|^{-\beta} e^{itD^a} f \|_{L_t^\infty L_{|x|}^2 L_\omega^2} \lesssim \| e^{itD^a} f \|_{L_t^\infty \dot{H}^\beta} \lesssim \| f \|_{\dot{H}^\beta} \simeq \| 2^{j\beta} r^{\frac{n-1}{2}} f_{jk} \|_{\ell_k^2 \ell_j^2 L_r^2}$$

for $\beta \in [0, \frac{n}{2})$, where we have used Littlewood-Paley-Stein decomposition for f in the last inequality. Also we can rewrite the weighted Strichartz estimates (1.22) as

$$(5.13) \quad \begin{aligned} \| |x|^{\frac{n}{2}-\frac{n}{p}-\gamma} e^{itD^a} f(x) \|_{L_{t,|x|}^p L_\omega^2} &\lesssim \| D^{\gamma-\frac{a}{p}} \Lambda_\omega^{\frac{1}{2}-\gamma} f \|_{L_x^2} \\ &= \| 2^{k(\frac{1}{2}-\gamma)} 2^{j(\gamma-\frac{a}{p})} r^{\frac{n-1}{2}} f_{jk} \|_{\ell_k^2 \ell_j^2 L_r^2}, \end{aligned}$$

if $\gamma \in (\frac{1}{2}, \frac{n}{2})$, $a > 0$ and $p \in [2, \infty]$.

Now for fixed $q \geq r \geq 2$, if we set $\theta = 1 - 2(\frac{1}{r} - \frac{1}{q})$, $p = q - \frac{2q}{r} + 2$, by complex interpolation between (5.12) and (5.13) and using Theorem 5.6.3 in [1], we get

$$\begin{aligned} \| |x|^{-\alpha} e^{itD^a} f(x) \|_{L_t^q L_{|x|}^r L_\omega^2} &\lesssim \| f_{jk} \|_{\ell_{2,k}^{(\frac{1}{2}-\gamma)\theta} ((\ell_{2,j}^\beta L_r^2 d\mu, \ell_{2,j}^{\gamma-\frac{a}{p}} L_r^2 d\mu)_{[\theta]})} \\ &= \| f_{jk} \|_{\ell_{2,k}^b \ell_{2,j}^s L_r^2} \\ &= \| 2^{kb} 2^{js} r^{\frac{n-1}{2}} f_{jk} \|_{\ell_k^2 \ell_j^2 L_r^2} \\ &= \| f \|_{\dot{H}_\omega^{s,b}}, \end{aligned}$$

where $d\mu = r^{n-1} dr$, and

$$\begin{aligned} -\alpha + \frac{a}{q} + \frac{n}{r} &= -s + \frac{n}{2}, \\ 2 \leq r \leq q, \quad \frac{1}{q} - \frac{n-1}{2} + \frac{n-1}{r} &< \alpha < \frac{n}{r}, \\ \begin{cases} b \geq -\alpha + \frac{1}{q} - (n-1)(\frac{1}{2} - \frac{1}{r}), & \text{if } \frac{1}{q} - \frac{n-1}{2} + \frac{n-1}{r} < \alpha < \frac{n}{r}, \\ b > -\frac{n-1}{q} - (n-1)(\frac{1}{2} - \frac{1}{r}), & \text{if } \frac{n}{r} \leq \alpha < \frac{n}{r}. \end{cases} \end{aligned}$$

6. AN APPLICATION: STRAUSS CONJECTURE WHEN $n = 2, 3$

As an application of the generalized Strichartz estimates in Theorem 1.1, Theorem 1.3 and Theorem 1.4, we prove the Strauss conjecture with low regularity when $n = 2, 3$.

Let $n = 2, 3$, $s_c = \frac{n}{2} - \frac{2}{p-1}$ be the critical index of regularity, $s_d = \frac{1}{2} - \frac{1}{p}$, $p_{conf} = 1 + \frac{4}{n-1}$ be the conformal index, p_c be the solution of $s_c = s_d$ and $p > 1$. Let $F_p(u)$ be a function such that

$$(6.1) \quad |F_p(u)| \leq C|u|^p, \quad |F'_p(u)| \leq C|u|^{p-1}$$

for some $C > 0$ and $p > 1$, consider the following semilinear wave equations

$$(6.2) \quad \begin{cases} (\partial_t^2 - \Delta)u = F_p(u) \\ u(0, x) = f, \partial_t u(0, x) = g. \end{cases}$$

Strauss conjecture asserts that for $p > p_c$, the problem (6.2) has a global solution, when the initial data (f, g) is sufficiently small and smooth. This conjecture was verified by Georgiev, Lindblad and Sogge in [9] and Tataru [38] for smooth data. Then the remaining problem is the case of low regularity. There has been many partial results on this field.

When $p \geq p_{conf}$, we have the global result with small $(f, g) \in \dot{H}^{s_c} \times \dot{H}^{s_c-1}$ in Lindblad-Sogge [21]. When the initial data $(f, g) \in \dot{H}_{rad}^{s_c} \times \dot{H}_{rad}^{s_c-1}$ (here we use the subscript “rad” to emphasize that the function is radial), the results are known when $s_c > \frac{1}{2n}$ or $n \leq 4$, see Lindblad-Sogge [21], Sogge [29] and Hidano [10]. Then Fang-Wang [5] and Hidano-Metcalf-Smith-Sogge-Zhou [13] proved the case with $2 \leq n \leq 4$ for small initial data $(f, g) \in \dot{H}_{\omega}^{s_c, b} \times \dot{H}_{\omega}^{s_c-1, b}$ with certain b , by proving certain $|x|$ -weighted Strichartz estimates like (1.22). When $2 < p < p_c$ and $n = 2, 3$, the radial local results with low regularity s_d have been obtained in Theorem 4.1 of Sogge [29] ($n = 3$) and Hidano-Kurokawa [11] ($n = 2$).

Now we present our results on Strauss conjecture.

Theorem 6.1. *Let $n = 2, 3$,*

$$\frac{1}{q} = \begin{cases} \frac{2}{p-1} - (n-2) & p_c < p < p_{conf} , \\ \frac{1}{p} + \frac{3-n}{2} & 2 < p < p_c , \end{cases}$$

$D^{s, b} = \dot{H}_{\omega}^{s, b} \times \dot{H}_{\omega}^{s-1, b}$, and $S = L_t^{qp} L_{|x|}^p L_{\omega}^{\infty}$ be the solution space. If $p \in (p_c, p_{conf})$, and $(f, g) \in D^{s_c, b}$ with norm bounded by $\epsilon \ll 1$ and $b > \frac{1}{qp} + \frac{n-1}{p}$, then there is a unique global weak solution $u \in S$ to (6.2). Also, if $p \in (2, p_c)$ and $(f, g) \in D^{s_d, b}$ with norm bounded by $\epsilon \ll 1$ and $b > \frac{1}{qp} + \frac{n-1}{p}$, we can have a solution $u \in S_{T_{\epsilon}}$ with $T_{\epsilon} = c\epsilon^{\frac{1}{s_c-s_d}}$ and $c \ll 1$.

Remark 6.1. The lifespan given in this result is essentially sharp. In [42] and [43], Zhou proved that the life span T_{ϵ} of classical solutions to the equation (6.2) has order $\epsilon^{\frac{1}{s_c-s_d}}$ when $n = 2, 3$ and $2 < p < p_c$ (see also Lindblad [20] for the case $n = 3$).

Remark 6.2. The choice of b can be possibly improved, if we apply Theorem 1.2 and 1.4.

Remark 6.3. Theorem 1.8 can also be employed to give an alternative proof of Theorem 6.1 for $2 < p < p_c$. See Yu [41] for the related argument for $n = 3$.

For the endpoint case $p = p_c$, we can also get the almost global result when $n = 2$. Let $X^b = H_{\omega}^{s_c+\delta, b} \times H_{\omega}^{s_c-1+\delta, b}$ with $\delta > 0$. Then we have

Theorem 6.2. *Let $n = 2$, $q = \frac{p-1}{2}$, $p = p_c$, $S = L_T^{qp} L_{|x|}^p L_\omega^\infty$ and $b > \frac{1}{qp} + \frac{1}{p}$. Suppose that $(f, g) \in X^b$ with norm bounded by $\epsilon \ll 1$, then there is a unique almost global weak solution $u \in S_{T_\epsilon}$ to (6.2), with $T_\epsilon = \exp(c\epsilon^{-(p-1)^2/2})$ and $c \ll 1$.*

Remark 6.4. The sharp lifespan has been proven to be $T_\epsilon = \exp(C\epsilon^{-p(p-1)})$ for $n \leq 3$ (see Zhou [43]). For the higher dimension $n \leq 8$, Lindblad and Sogge [22] proved the existence results with $T_\epsilon = \exp(C\epsilon^{-p(p-1)})$ (together with the radial results for any dimension $n \geq 3$ with same lifespan). In [38], Tataru proved the almost global result with $T_\epsilon = \exp(C\epsilon^{-r})$ with $r = (p_c^2 - 1)/(3p_c + 1)$ for small smooth data and all dimensions.

In this section, we give a proof of the Strauss conjecture with low regularity when the dimension is 2, 3, i.e., Theorem 6.1, by using essentially only the generalized Strichartz estimates. Moreover, we can also prove the almost global result Theorem 6.2 for the endpoint case $p = p_c$ and $n = 2$, by using local in time generalized Strichartz estimates.

6.1. Global Results for $p > p_c$. When $p \in (p_c, p_{conf})$, we have $\frac{1}{q} = \frac{2}{p-1} - (n-2)$.

If $u \in S$, we define Πu to be the solution of the wave equation

$$\square \Pi u = F_p(u)$$

with initial data $(f, g) \in D^{s_c, b}$. Then it suffices to show that when the initial data is small enough in $D^{s_c, b}$, then $\Pi : S \rightarrow S$ and the map is a contraction map on small balls of S .

Now we prove this claim. First, note that $\Pi u = v_h + v_i$ with v_h being the solution to the homogeneous equation with initial data $(f, g) \in D^{s_c, b}$, and v_i the solution to the inhomogeneous equation with null initial data $(0, 0)$.

If $n = 2$, u is radial and $p_c < p < p_{conf}$, then (qp, p, s) and $(q', \infty, 1-s)$ both satisfy the conditions in Theorem 1.4, i.e.,

$$\frac{1}{qp} + \frac{1}{p} < \frac{1}{2} \text{ and } q < 2.$$

Thus by Christ-Kiselev lemma [3], we have for $n = 2$

$$(6.3) \quad \|v_i\|_S \lesssim \|F_p(u)\|_{L_t^q L_{rad}^1}.$$

Moreover, if $n = 3$, Sogge (Theorem 4.2 of [29]) proves the same radial inhomogeneous inequality (6.3). Then by the comparison principle for the wave equation with $n = 2, 3$, we have

$$\|v_i\|_S \lesssim \|F_p(u)\|_{L_t^q L_{|x|}^1 L_\omega^\infty} \lesssim \|u\|_S^p.$$

Since $(f, g) \in D^{s_c, b}$ with $b > \frac{1}{pq} + \frac{n-1}{p}$, an application of Theorem 1.1 and Sobolev embedding on the sphere yields that

$$\|v_h\|_S \lesssim \|(f, g)\|_{D^{s_c, b}}.$$

Thus we know that

$$(6.4) \quad \|\Pi u\|_S \lesssim \|(f, g)\|_{D^{s_c, b}} + \|u\|_S^p.$$

From this inequality, our claim follows immediately (recall (6.1)).

6.2. Local Results for $p \in (2, p_c)$. In this subsection, we prove the local results in Theorem 6.1 when $p \in (2, p_c)$. Define $\frac{1}{q} = \frac{1}{p} + \frac{3-n}{2}$, $D^{s_d, b} = \dot{H}_{\omega}^{s_d, b} \times \dot{H}_{\omega}^{s_d-1, b}$ and let $S_{T_\epsilon} = L_{T_\epsilon}^{qp} L_{|x|}^p L_{\omega}^{\infty}$ be the solution space with $T_\epsilon = c\epsilon^{\frac{1}{s_c - s_d}}$, and let ϵ be the norm of the data in $D^{s_d, b}$. We want to prove that the map Π is a contraction map on small balls of S_{T_ϵ} .

If $n = 2$ and u is radial, let $s = s_d = 1/2 - 1/p$, then (qp, p, s) and $(q', \infty, 1-s)$ satisfy the condition for (1.14) and (1.13). Thus by Christ-Kiselev lemma [3], we have for $n = 2$

$$(6.5) \quad \|v_i\|_{S_T} \leq CT^{\frac{1}{qp} + \frac{n-1}{p} - \frac{n-1}{2}} \|F_p(u)\|_{L_T^q L_{rad}^1}.$$

Moreover, if $n = 3$, Sogge (Theorem 4.2 in [29]) proves the same radial inhomogeneous inequality (6.5). Then by the comparison principle for wave equation with $n = 2, 3$, we have

$$\|v_i\|_{S_T} \leq CT^{\frac{1}{qp} + \frac{n-1}{p} - \frac{n-1}{2}} \|F_p(u)\|_{L_T^q L_{|x|}^1 L_{\omega}^{\infty}} \leq CT^{\frac{1}{qp} + \frac{n-1}{p} - \frac{n-1}{2}} \|u\|_{S_T}^p.$$

Since $(f, g) \in D^{s_d, b}$ with $b > \frac{1}{qp} + \frac{1}{p}$ and norm ϵ , an application of Theorem 1.3 yields that

$$\|v_h\|_{S_T} \leq CT^{\frac{1}{qp} + \frac{n-1}{p} - \frac{n-1}{2}} \|(f, g)\|_{D^{s_d, b}} \leq CT^{\frac{1}{qp} + \frac{n-1}{p} - \frac{n-1}{2}} \epsilon.$$

Thus we know that

$$(6.6) \quad \|\Pi u\|_{S_T} \leq CT^{\frac{1}{qp} + \frac{n-1}{p} - \frac{n-1}{2}} \epsilon + CT^{\frac{1}{qp} + \frac{n-1}{p} - \frac{n-1}{2}} \|u\|_{S_T}^p.$$

Moreover, we have (recall (6.1))

$$(6.7) \quad \|\Pi u - \Pi v\|_{S_T} \leq CT^{\frac{1}{qp} + \frac{n-1}{p} - \frac{n-1}{2}} (\|u\|_{S_T} + \|v\|_{S_T})^{p-1} \|u - v\|_{S_T}.$$

Thus if we choose $T_\epsilon = c\epsilon^{\frac{1}{s_c - s_d}}$ with c sufficiently small, we can see from (6.6) and (6.7) that Π is a contraction map on the complete set

$$\{u \in S_{T_\epsilon} \mid \|u\|_{S_{T_\epsilon}} \leq 2CT^{\frac{1}{qp} + \frac{n-1}{p} - \frac{n-1}{2}} \epsilon \simeq \epsilon^{\frac{1}{p}}\},$$

which completes the proof of Theorem 6.1.

6.3. Almost Global Results for $p = p_c$ and $n = 2$. In this subsection, we prove the almost global results in Theorem 6.2. Recall that when $p = p_c$ and $n = 2$, we have the local in time estimates (1.10).

We use a similar argument to prove this result. Since $(f, g) \in X^b$ with $b > \frac{1}{qp} + \frac{1}{p}$, an application of Theorem 1.3 yields that

$$\|v_h\|_{S_{T_\epsilon}} \lesssim (\ln(2 + T_\epsilon))^{1/qp} \|(f, g)\|_{X^b} \lesssim (\ln(2 + T_\epsilon))^{1/qp} \epsilon.$$

Similarly, we have

$$\|v_i\|_{S_{T_\epsilon}} \lesssim (\ln(2 + T_\epsilon))^{1/qp} \|F_p(u)\|_{L_t^q L_{|x|}^1 L_{\omega}^{\infty}} \lesssim (\ln(2 + T_\epsilon))^{1/qp} \|u\|_{S_{T_\epsilon}}^p.$$

By Theorem 1.4 in the radial case and the comparison principle.

Now if we choose $T_\epsilon = \exp(c\epsilon^{-q(p-1)})$ such that

$$\|v_h\|_{S_{T_\epsilon}} \leq \epsilon^{1/p}.$$

Then if $\|u\|_{S_{T_\epsilon}} \leq 2\epsilon^{1/p}$, we have

$$\|\Pi u\|_{S_{T_\epsilon}} \leq \epsilon^{1/p} + (\epsilon^{1/p})^p \epsilon^{-(p-1)/p} \leq 2\epsilon^{1/p}.$$

Moreover, if $u, v \in S_{T_\epsilon}$ with norm bounded by $2\epsilon^{1/p}$, then

$$\|\Pi(u - v)\|_{S_{T_\epsilon}} \leq C(\ln(2 + T_\epsilon))^{1/qp} (4\epsilon^{1/p})^{p-1} \|u - v\|_{S_{T_\epsilon}} \leq \frac{1}{2} \|u - v\|_{S_{T_\epsilon}},$$

where we have used the assumption (6.1). Thus the map Π is a contraction map on S_{T_ϵ} with norm bounded by $2\epsilon^{1/p}$. This completes the proof of Theorem 6.2.

7. APPENDIX

7.1. Sobolev embedding on the sphere \mathbb{S}^{n-1} . We give a simple proof of the Sobolev inequalities used in of Section 2.1 and 5.2. These inequalities should be true in general. For the sake of completeness, we give a proof here.

Lemma 7.1 (Sobolev embedding I). *Let $2 \leq q < \infty$, we have the following*

$$(7.1) \quad \|f\|_{L^q(\mathbb{S}^{n-1})} \leq C \|\Lambda_\omega^{\sigma(q)} f\|_{L^2(\mathbb{S}^{n-1})},$$

where

$$\sigma(q) = (n-1)\left(\frac{1}{2} - \frac{1}{q}\right).$$

Proof. Our proof is based on the spectral cluster estimates on the sphere (see e.g. [28] Lemma 4.2.4 page 129). Recall that if we define the spectral cluster operator

$$\chi_\lambda f = \sum_{\lambda_k \in [\lambda, \lambda+1]} E_k f$$

where E_k is the projection onto the one dimensional eigenspace with eigenvalue λ_k , then we have

$$(7.2) \quad \|\chi_\lambda f\|_{L^\infty(\mathbb{S}^{n-1})} \leq C(1 + \lambda)^{(n-2)/2} \|\chi_\lambda f\|_{L^2(\mathbb{S}^{n-1})}.$$

Thus for the Littlewood-Paley projector S_λ on the spectral interval $[\lambda, 2\lambda]$, by using the Cauchy-Schwartz inequality in j with $j \in \mathbb{Z}$, we have

$$(7.3) \quad \|S_\lambda f\|_{L^\infty} \leq C(1 + \lambda)^{(n-2)/2} \sum_{j \in [\lambda, 2\lambda]} \|\chi_j f\|_{L^2} \leq C(1 + \lambda)^{(n-1)/2} \|S_\lambda f\|_{L^2}.$$

Then by Hölder's inequality, we have

$$(7.4) \quad \|S_\lambda f\|_{L^q} \leq \|S_\lambda f\|_{L^\infty}^{1-2/q} \|S_\lambda f\|_{L^2}^{2/q} \leq C(1 + \lambda)^{(n-1)(1/2-1/q)} \|S_\lambda f\|_{L^2}$$

for any $2 \leq q \leq \infty$.

Finally, by the Littlewood-Paley-Stein theorem (see Theorem 2 [34]) for the sphere, we have

$$(7.5) \quad \|f\|_{L^q} \sim \|S_\lambda f\|_{L^q \ell_\lambda^2} \leq \|S_\lambda f\|_{\ell_\lambda^2 L^q} \leq C \|f\|_{H^{\sigma(q)}}$$

for any $2 \leq q < \infty$, which completes the proof. ■

Lemma 7.2 (Sobolev embedding II). *Let $p \geq 2$, then for the Littlewood-Paley projector S_λ on the spectral interval $[\lambda, 2\lambda]$ we have the following*

$$(7.6) \quad \|S_\lambda f\|_{L^\infty(\mathbb{S}^{n-1})} \leq C(1 + \lambda)^{(n-1)/p} \|f\|_{L^p(\mathbb{S}^{n-1})} .$$

Proof. Since the estimate for $\lambda \lesssim 1$ is trivial, we assume that $\lambda \gg 1$. Note that it is equivalent to prove the dual estimate of (7.6),

$$\|S_\lambda f\|_{L^{p'}(\mathbb{S}^{n-1})} \leq C\lambda^{(n-1)/p} \|f\|_{L^1(\mathbb{S}^{n-1})}$$

which is a consequence of the interpolation between the dual of (7.3), which says

$$\|S_\lambda f\|_{L^2(\mathbb{S}^{n-1})} \leq C\lambda^{(n-1)/2} \|f\|_{L^1(\mathbb{S}^{n-1})}$$

and

$$(7.7) \quad \|S_\lambda f\|_{L^1(\mathbb{S}^{n-1})} \leq C\|f\|_{L^1(\mathbb{S}^{n-1})} .$$

Hence we need only to prove (7.7). Let $P = \sqrt{-\Delta_\omega}$ and $\beta \in C_0^\infty$ be an even function on \mathbb{R} with support in $\pm(1, 2)$. Then

$$S_\lambda f = \beta^2(P/\lambda)f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \lambda \widehat{\beta}(\lambda t) e^{itP} \beta(P/\lambda) f(x) dt.$$

Note that proving (7.7) is equivalent to considering

$$T_\lambda(P)f(x) = \int_{\mathbb{R}} \lambda \widehat{\beta}(\lambda t) \cos(tP) \beta(P/\lambda) f(x) dt,$$

and proving

$$(7.8) \quad \|T_\lambda(P)f\|_{L^1(\mathbb{S}^{n-1})} \leq C\|f\|_{L^1(\mathbb{S}^{n-1})} .$$

Here

$$\cos tP f(x) = \sum_{k=1}^{\infty} \cos t\lambda_k E_k(f)(x) = u(t, x)$$

is the cosine transform of f . It is the solution of the wave equation

$$(\partial_t^2 - \Delta_\omega)u = 0, \quad u(0, \cdot) = f, \quad u_t(0, \cdot) = 0.$$

In order to prove (7.8), we shall use the finite propagation speed for solutions to the wave equation. Specifically, if f is supported in a geodesic ball $B(x_0, R)$ centered at x_0 with radius R , then $x \rightarrow \cos tP f$ vanishes outside of $B(x_0, R + T)$ if $0 \leq t \leq T$.

Let $1 = \eta(t) + \sum_{j=1}^{\infty} \rho(2^{-j}t)$ be a Littlewood-Paley partition of \mathbb{R} . Write $T_\lambda = T_\lambda^0 + \sum_{j \geq 1} T_\lambda^j$, where

$$(7.9) \quad T_\lambda^0(P)f = \int_{\mathbb{R}} \eta(\lambda t) \lambda \widehat{\beta}(\lambda t) \cos(tP) \beta(P/\lambda) f dt$$

and

$$(7.10) \quad T_\lambda^j(P)f = \int_{\mathbb{R}} \rho(2^{-j}\lambda t) \lambda \widehat{\beta}(\lambda t) \cos(tP) \beta(P/\lambda) f dt$$

We will prove $T_\lambda(P)$ satisfies (7.8) by showing $T_\lambda^0(P)$ and $\sum_{j \geq 1} T_\lambda^j(P)$ both satisfy (7.8).

Now

$$\begin{aligned}
T_\lambda^0(P)f(x) &= \int_{\mathbb{R}} \eta(\lambda t) \lambda \widehat{\beta}(\lambda t) \cos(tP) \beta(P/\lambda) f(x) dt \\
&= \int_{\mathbb{R}} \eta(\lambda t) \lambda \widehat{\beta}(\lambda t) \sum_{\lambda \leq \lambda_k \leq 2\lambda} \cos(t\lambda_k) \beta(\lambda_k/\lambda) e_k(x) \int_{\mathbb{S}^{n-1}} e_k(y) f(y) dy dt \\
&= \int_{\mathbb{S}^{n-1}} \left\{ \int_{\mathbb{R}} \eta(\lambda t) \lambda \widehat{\beta}(\lambda t) \sum_{\lambda \leq \lambda_k \leq 2\lambda} \cos(t\lambda_k) \beta(\lambda_k/\lambda) e_k(x) e_k(y) dt \right\} f(y) dy \\
&= \int_{\mathbb{S}^{n-1}} K_\lambda^0(x, y) f(y) dy
\end{aligned}$$

The finite propagation speed of the wave equation mentioned before implies that the kernel $K_\lambda^0(x, y)$ must satisfy

$$K_\lambda^0(x, y) = 0 \quad \text{if} \quad \text{dist}(x, y) > 8\lambda^{-1},$$

since $\cos tP$ will have a kernel that vanishes on this set when t belongs to the support of the integral defining $K_\lambda^0(x, y)$. Because of this, in order to prove T_λ^0 satisfies (7.8), it suffices to show that for all geodesic balls $B_{\lambda,0}$ with radius $8\lambda^{-1}$ one has the bound

$$(7.11) \quad \|T_\lambda^0 f\|_{L^1(B_{\lambda,0})} \leq C \|f\|_{L^1(\mathbb{S}^{n-1})},$$

for the L^1 norm over $B_{\lambda,0}$.

By using the Cauchy-Schwartz inequality, the dual of (7.2), and orthogonality, we can deduce that

$$\begin{aligned}
(7.12) \quad \|T_\lambda^0 f\|_{L^1(B_{\lambda,0})} &\leq C \lambda^{-(n-1)/2} \|T_\lambda^0 f\|_{L^2(\mathbb{S}^{n-1})} \\
&\leq C \lambda^{-(n-1)/2} \left(\sum_{l=\lambda}^{2\lambda} \|\chi_l f\|_{L^2(\mathbb{S}^{n-1})}^2 \right)^{1/2} \\
&\leq C \lambda^{-(n-1)/2} \lambda^{1/2} \lambda^{(n-2)/2} \|f\|_{L^1(\mathbb{S}^{n-1})} \\
&\leq C \|f\|_{L^1(\mathbb{S}^{n-1})}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
(7.13) \quad T_\lambda^j f(x) &= \int_{\mathbb{R}} \rho(2^{-j}\lambda t) \lambda \widehat{\beta}(\lambda t) \cos(tP) \beta(P/\lambda) f(x) dt \\
&= \int_{\mathbb{S}^{n-1}} \left\{ \int_{\mathbb{R}} \rho(2^{-j}\lambda t) \lambda \widehat{\beta}(\lambda t) \sum_{\lambda \leq \lambda_k \leq 2\lambda} \cos(t\lambda_k) \beta(\lambda_k/\lambda) e_k(x) e_k(y) dt \right\} f(y) dy \\
&= \int_{\mathbb{S}^{n-1}} K_\lambda^j(x, y) f(y) dy
\end{aligned}$$

has the property that $K_\lambda^j(x, y) = 0$ if $\text{dist}(x, y) \geq 8 \cdot 2^{j+1} \cdot \lambda^{-1}$. Note that the dyadic cutoff localizes to $|t| \approx \lambda^{-1} 2^j$. Hence the arguments in (7.12) yields the bound $(2^j \lambda^{-1})^{(n-1)/2} \lambda^{1/2} (2^j)^{-N} (\lambda^{(n-2)/2}) \|f\|_{L^1}$, if N is a large enough integer. Here the term $2^j \lambda^{-1}$ comes from the volume of geodesic ball $B_{\lambda,j}$ with radius $8 \cdot 2^{j+1} \cdot \lambda^{-1}$, and $(2^j)^{-N}$ comes from $\beta \in \mathcal{S}$. Thus we have

$$\|T_\lambda^j f\|_{L^1} \lesssim 2^{-jN} \|f\|_{L^1}.$$

This forms a geometric series and thus the sum over $j = 1, \dots, \infty$ terms enjoys the property (7.8). ■

7.2. Morawetz-KSS estimates. The Morawetz-KSS estimates are in fact an easy consequence of the energy estimate and the local energy estimate. Recall that the local energy estimate can be stated as follows (see e.g. (1.10) in [5])

$$(7.14) \quad R^{-\frac{1}{2}} \|e^{itD} f\|_{L^2_{t,x:|x|\leq R}} \lesssim \|f\|_{L^2_x}.$$

Recall also that we have the energy estimate

$$T^{-1/2} \|e^{itD} f\|_{L^2_{t\in[0,T]} L^2_x} \leq \|e^{itD} f\|_{L^\infty_t L^2_x} \leq \|f\|_{L^2_x}.$$

The Morawetz estimates with $\mu > 1/2$ can be proven as follows,

$$\|\langle x \rangle^{-\mu} e^{itD} f\|_{L^2_{t,x}} \lesssim \sum_{j \geq 0} 2^{-j\mu} \|e^{itD} f\|_{L^2_{t,x:|x|\leq 2^j}} \lesssim \sum_{j \geq 0} 2^{-j(\mu-1/2)} \|f\|_{L^2_x} \lesssim \|f\|_{L^2_x}.$$

Now we deal with the case $\mu \leq 1/2$. We consider first the case when $T \lesssim 1$. In this case, the Morawetz-KSS estimates are in fact weaker than the energy estimate,

$$\|\langle x \rangle^{-\mu} e^{itD} f\|_{L^2_{[0,T]} L^2_x} \lesssim T^{\frac{1}{2}} \|f\|_{L^2_x} \lesssim A_\mu(T) \|f\|_{L^2_x}.$$

For the remaining case with $T \geq 2$, we use the energy estimate to deal with the region $|x| \geq T$,

$$\|\langle x \rangle^{-\mu} e^{itD} f\|_{L^2_{[0,T]} L^2_{x:|x|\geq T}} \lesssim T^{-\mu} \|e^{itD} f\|_{L^2_{[0,T]} L^2_x} \lesssim T^{\frac{1}{2}-\mu} \|f\|_{L^2_x} \lesssim A_\mu(T) \|f\|_{L^2_x}.$$

For the remaining region, we use instead (7.14)

$$\begin{aligned} \|\langle x \rangle^{-\mu} e^{itD} f\|_{L^2_{[0,T]} L^2_x}^2 &\lesssim \sum_{0 \leq j \lesssim \ln T} 2^{-2j\mu} \|e^{itD} f\|_{L^2_T L^2_{|x|\leq 2^j}}^2 \\ &\lesssim \sum_{0 \leq j \lesssim \ln T} 2^{j(1-2\mu)} \|f\|_{L^2_x}^2 \\ &\lesssim A_\mu(T)^2 \|f\|_{L^2_x}^2. \end{aligned}$$

This completes the proof of (2.7).

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